A Diagrammer's Note on Superconducting Fluctuation Transport for Beginners: I. Conductivity and Thermopower

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Abstract

A diagrammatic approach based on thermal Green function to superconducting fluctuation transport is reviewed keeping consistency with Ginzburg-Landau theory. The correct expression of the heat current vertex for Cooper pairs is clarified via Jonson-Mahan formula and Ward identities.

1 Introduction

In this Note, I, a diagrammer, try to supply you, beginners, with some detailed calculations usually skipped in the references written by professionals.[1] I hope that you can save time and efforts by this note and reserve enough energy to go farther.

This note is intended to be a primer[2] to the textbook: *Theory of Fluctuations in Superconductors* [2, 3]. It is the first of the series consisting of three notes. This first note mainly discusses the transport in the absence of magnetic field. The second and the third mainly discuss the transport under magnetic field.

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[1] I shall remark on some mistakes made by professionals in several footnotes. Such notes might be helpful for you to avoid unnecessary struggle with the mistakes.

[2] In the discussion of superconducting fluctuation transport $(T > T_c)$ I only consider the Aslamazov-Larkin (AL) contribution and leave the consideration of the Maki-Thompson (MT) and the density-of-states (DOS) contributions to the textbook [2, 3]. The AL transport is equivalent to the time-dependent Ginzburg-Landau (GL) transport.

I expect that you have some knowledge of the Feynman-diagram technique at finite temperature and the linear response calculation via the thermal Green function. I will skip detailed explanations of these in the following.

We use the unit of $\hbar = k_B = c = 1$.

2   Hamiltonian

In grand canonical formalism we use K

$$K = H - \mu N, \tag{1}$$

instead of the Hamiltonian H where N and μ are the number and chemical potential of electrons. We decompose K as

$$K = K_0 + V, \tag{2}$$

where

$$K_0 = \sum_{\mathbf{p}} \xi_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}} \right), \tag{3}$$

is the kinetic energy measured from the chemical potential μ with

$$\xi_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} - \mu, \tag{4}$$

and

$$V = -g \sum_{\mathbf{q}} P_{\mathbf{q}}^\dagger P_{\mathbf{q}}, \tag{5}$$

is the local attractive interaction with g being a positive constant. Here $a_{\mathbf{p}}^\dagger$ represents the creation operator of an up-spin electron with momentum $\mathbf{p}$, $b_{\mathbf{p}}$ represents the annihilation operator of a down-spin electron and so on. The creation and annihilation operators of Cooper Pairs are

$$P_{\mathbf{q}}^\dagger = \sum_{\mathbf{p}} a_{\mathbf{p}+\frac{\mathbf{q}}{2}}^\dagger b_{-\mathbf{p}+\frac{\mathbf{q}}{2}}^\dagger, \qquad P_{\mathbf{q}} = \sum_{\mathbf{p}} b_{-\mathbf{p}+\frac{\mathbf{q}}{2}} a_{\mathbf{p}+\frac{\mathbf{q}}{2}}. \tag{6}$$

The (imaginary) time dependence of the operator A is given as

$$A(\tau) = e^{K\tau} A e^{-K\tau}, \tag{7}$$

or

$$\frac{\partial}{\partial \tau} A(\tau) = \left[K, A(\tau) \right]. \tag{8}$$

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3 Electron Propagator

The propagator for electrons $G(\mathbf{p}, \tau)$ is defined as

$$G(\mathbf{p}, \tau) = -\langle T_\tau\{a_\mathbf{p}(\tau)a_\mathbf{p}^\dagger\}\rangle, \tag{9}$$

where $\langle X \rangle$ is the expectation value of X using the density matrix defined by K. In the following, throughout the series of three notes, the Zeeman splitting is neglected so that

$$-\langle T_\tau\{b_\mathbf{p}(\tau)b_\mathbf{p}^\dagger\}\rangle = G(\mathbf{p}, \tau). \tag{10}$$

The operation of the (imaginary) time ordering T_τ is explicitly expressed as

$$G(\mathbf{p}, \tau) = -\theta(\tau)\langle a_\mathbf{p}(\tau)a_\mathbf{p}^\dagger\rangle + \theta(-\tau)\langle a_\mathbf{p}^\dagger a_\mathbf{p}(\tau)\rangle, \tag{11}$$

for electrons where $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. Since the time evolution of the free electron is given as[3]

$$a_\mathbf{p}(\tau) = e^{-\xi_\mathbf{p}\tau}a_\mathbf{p}, \tag{12}$$

the propagator for free electrons $G_0(\mathbf{p}, \tau)$ becomes[4]

$$G_0(\mathbf{p}, \tau) = e^{-\xi_\mathbf{p}\tau}\{-\theta(\tau)[1 - f(\xi_\mathbf{p})] + \theta(-\tau)f(\xi_\mathbf{p})\}, \tag{13}$$

where $f(x)$ is the Fermi distribution function

$$f(x) = \frac{1}{e^{\beta x} + 1}. \tag{14}$$

The full propagator $G(\mathbf{p}, \tau)$ is expressed as the superposition of this-type of function:

$$G(\mathbf{p}, \tau) = \int_{-\infty}^{\infty} d\epsilon \rho(\epsilon) e^{-\epsilon\tau}\{-\theta(\tau)[1 - f(\epsilon)] + \theta(-\tau)f(\epsilon)\}. \tag{15}$$

Here $\rho(\epsilon)$ is the spectral function.

The Fourier transform of the propagator is defined as

$$G(\mathbf{p}, \tau) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty} G(\mathbf{p}, i\varepsilon_n)e^{-i\varepsilon_n\tau}, \tag{16}$$

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[3] $a_\mathbf{p}^\dagger(\tau) = e^{\xi_\mathbf{p}\tau}a_\mathbf{p}^\dagger$.

[4] $a_\mathbf{p}a_\mathbf{p}^\dagger = 1 - a_\mathbf{p}^\dagger a_\mathbf{p}$. For free electrons $\langle a_\mathbf{p}^\dagger a_\mathbf{p}\rangle_0 = f(\xi_\mathbf{p})$ where $\langle X \rangle_0$ is the expectation value of X using the density matrix defined by K_0.

$$G(\mathbf{p}, i\varepsilon_n) = \int_0^\beta d\tau G(\mathbf{p}, \tau) e^{i\varepsilon_n \tau}, \tag{17}$$

where $\varepsilon_n = (2n + 1)\pi T$ with n being integer. Here T is the temperature and β is the inverse temperature $\beta = 1/T$.

The free propagator is transformed as

$$G_0(\mathbf{p}, i\varepsilon_n) = -\int_0^\beta d\tau \langle a_\mathbf{p}(\tau) a_\mathbf{p}^\dagger \rangle e^{i\varepsilon_n \tau} = -\left[1 - f(\xi_\mathbf{p})\right] \int_0^\beta d\tau e^{(i\varepsilon_n - \xi_\mathbf{p})\tau}. \tag{18}$$

Using $e^{i\varepsilon_n \beta} = -1$, we obtain

$$G_0(\mathbf{p}, i\varepsilon_n) = \frac{1}{i\varepsilon_n - \xi_\mathbf{p}}. \tag{19}$$

The full propagator $G(\mathbf{p}, i\varepsilon_n)$ is expressed as the superposition of this-type of function:

$$G(\mathbf{p}, i\varepsilon_n) = \int_{-\infty}^\infty d\epsilon \frac{\rho(\epsilon)}{i\varepsilon_n - \epsilon}. \tag{20}$$

The retarded and advanced propagators, $G^R(\mathbf{p}, \varepsilon)$ and $G^A(\mathbf{p}, \varepsilon)$, are obtained from the thermal propagator $G(\mathbf{p}, i\varepsilon_n)$ as

$$G^R(\mathbf{p}, \varepsilon) = G(\mathbf{p}, \varepsilon + i\delta), \qquad G^A(\mathbf{p}, \varepsilon) = G(\mathbf{p}, \varepsilon - i\delta), \tag{21}$$

where $\delta = +0$. The spectral function $\rho(\varepsilon)$ is related[5] to the imaginary part of the retarded propagator as

$$\mathrm{Im} G^R(\mathbf{p}, \varepsilon) = -\pi \rho(\varepsilon). \tag{22}$$

The full and free propagators are related by the Dyson equation

$$G(\mathbf{p}, i\varepsilon_n)^{-1} = G_0(\mathbf{p}, i\varepsilon_n)^{-1} - \Sigma(\mathbf{p}, i\varepsilon_n), \tag{23}$$

where $\Sigma(\mathbf{p}, i\varepsilon_n)$ is the self-energy.

4   Cooper-Pair Propagator

The effective interaction for Cooper pairs in the ladder approximation $L(\mathbf{q}, i\omega_m)$ is given by

$$\begin{aligned} -L(\mathbf{q}, i\omega_m) &= g + g\Pi(\mathbf{q}, i\omega_m)g + g\Pi(\mathbf{q}, i\omega_m)g\Pi(\mathbf{q}, i\omega_m)g + \cdots \\ &= \frac{g}{1 - g\Pi(\mathbf{q}, i\omega_m)}, \end{aligned} \tag{24}$$

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[5] We have employed the identity

$$\frac{1}{x + i\delta} = \frac{\mathrm{P}}{x} - i\pi\delta(x),$$

where P denotes that the principal value should be taken when it is integrated.
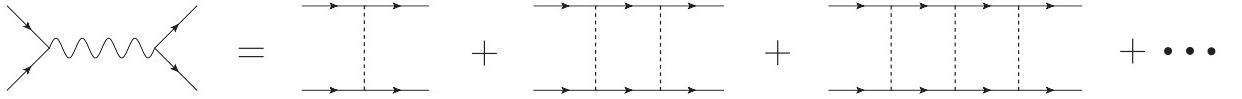
Figure 1: Effective interaction for Cooper pairs: The wavy and broken lines represent $-L$ and $-(-g)$. The solid line with arrow represents G_0. All the Feynman diagrams in this note are drawn by JaxoDraw.

where[6]

$$\Pi(\mathbf{q}, i\omega_m) = T \sum_n \sum_{\mathbf{p}} G_0(\mathbf{p} + \frac{\mathbf{q}}{2}, i\varepsilon_n) G_0(-\mathbf{p} + \frac{\mathbf{q}}{2}, i\omega_m - i\varepsilon_n), \qquad (25)$$

and $\omega_m = 2m\pi T$ with m being integer.

The propagator for Cooper pairs $D(\mathbf{q}, \tau)$ is defined as

$$D(\mathbf{q}, \tau) = -\langle T_\tau \{ P_{\mathbf{q}}(\tau) P_{\mathbf{q}}^\dagger \} \rangle, \qquad (26)$$

and its Fourier transform in the ladder approximation $D(\mathbf{q}, i\omega_m)$ is given by

$$\begin{aligned}
-D(\mathbf{q}, i\omega_m) &= \Pi(\mathbf{q}, i\omega_m) + \Pi(\mathbf{q}, i\omega_m)g\Pi(\mathbf{q}, i\omega_m) + \cdots \\
&= \frac{\Pi(\mathbf{q}, i\omega_m)}{1 - g\Pi(\mathbf{q}, i\omega_m)}.
\end{aligned} \qquad (27)$$

The relation[7] between $D(\mathbf{q}, i\omega_m)$ and $L(\mathbf{q}, i\omega_m)$ is

$$L(\mathbf{q}, i\omega_m) = -g + g^2 D(\mathbf{q}, i\omega_m). \qquad (28)$$

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[6] Since the particle-particle pair propagator $\Pi(\mathbf{q}, i\omega_m)$ depends on the shape of the region of $\mathbf{p}$-summation, we have chosen the symmetric configuration in $\mathbf{p}$-space as (25). On the other hand, the particle-hole pair propagator is free from the choice. See Fukuyama, Hasegawa and Narikiyo: J. Phys. Soc. Japan **60**, 2013 (1991) for these discussions. The imaginary part of $P^R(q, \omega)$ proportional to q^2 derived in Sergeev, Reizer and Mitin: Phys. Rev. B **66**, 104504 (2002) is absent in the symmetric choice.

[7] In the forthcoming discussion employing the Ward identity the inverse of the propagator $D(\mathbf{q}, i\omega_m)^{-1}$ plays the central role. In the limit of $T \to T_c$, $\mathbf{q} \to 0$ and $\omega_m \to 0$,

$$D(\mathbf{q}, i\omega_m)^{-1} = L(\mathbf{q}, i\omega_m)^{-1} \cdot \frac{g}{\Pi(\mathbf{q}, i\omega_m)} \fallingdotseq L(\mathbf{q}, i\omega_m)^{-1} \cdot \frac{g}{\Pi(0, 0)} \fallingdotseq L(\mathbf{q}, i\omega_m)^{-1} \cdot g^2,$$

where $1 - g\Pi(0, 0) \sim O(T - T_c)$ and $L(\mathbf{q}, i\omega_m)^{-1} \sim O(T - T_c, \mathbf{q}^2, \omega_m)$ as

$$L(\mathbf{q}, i\omega_m)^{-1} = -[g^{-1} - \Pi(0, 0)] - [\Pi(0, 0) - \Pi(\mathbf{q}, i\omega_m)].$$

Thus in this limit

$$D_\Delta(\mathbf{q}, i\omega_m)^{-1} \fallingdotseq L(\mathbf{q}, i\omega_m)^{-1}.$$

In terms of the propagator of the order-parameter $D_\Delta(\mathbf{q}, i\omega_m)$ this relation is rewritten as

$$L(\mathbf{q}, i\omega_m) = -g + D_\Delta(\mathbf{q}, i\omega_m), \qquad (29)$$

where the propagator is introduced as

$$D_\Delta(\mathbf{q}, \tau) = -\langle T_\tau \{\Delta_\mathbf{q}(\tau)\Delta_\mathbf{q}^\dagger\}\rangle, \qquad (30)$$

with

$$\Delta_\mathbf{q}^\dagger = g \cdot P_\mathbf{q}^\dagger, \qquad \Delta_\mathbf{q} = g \cdot P_\mathbf{q}. \qquad (31)$$

The retarded and advanced propagators, $D^R(\mathbf{q}, \omega)$ and $D^R(\mathbf{q}, \omega)$, are obtained from the thermal propagator $D(\mathbf{q}, i\omega_m)$ as

$$D^R(\mathbf{q}, \omega) = D(\mathbf{q}, \omega + i\delta), \qquad D^A(\mathbf{q}, \omega) = D(\mathbf{q}, \omega - i\delta). \qquad (32)$$

In the ladder approximation, the full and free propagators, $D(\mathbf{q}, i\omega_m)$ and $D_0(\mathbf{q}, i\omega_m)$, are related by the Dyson equation as

$$D(\mathbf{q}, i\omega_m)^{-1} = D_0(\mathbf{q}, i\omega_m)^{-1} - \Sigma(\mathbf{q}, i\omega_m), \qquad (33)$$

where

$$D_0(\mathbf{q}, i\omega_m) = -\Pi(\mathbf{q}, i\omega_m), \qquad \Sigma(\mathbf{q}, i\omega_m) = -g. \qquad (34)$$

In the limits of low energy and long wavelength the retarded effective interaction $L^R(\mathbf{q}, \omega) = L(\mathbf{q}, \omega + i\delta)$ obtained from (24) is[8]

$$L^R(\mathbf{q}, \omega) = -\frac{1}{N(0)} \frac{1}{\epsilon + \xi_0^2 \mathbf{q}^2 - i\omega\tau_0}, \qquad (35)$$

where

$$\epsilon \equiv \ln \frac{T}{T_c} \fallingdotseq \frac{T - T_c}{T_c}, \qquad (36)$$

and $N(0)$ is the density of states per spin at the chemical potential. This effective interaction has Ornstein-Zernike form in space-direction and Debye form in time direction. The functional form of (35) holds beyond the ladder approximation so that we regard ϵ, ξ_0 and τ_0 as phenomenological parameters in the following.

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[8] See §6.2 in [3] for the microscopic calculation in the ladder approximation. In §6.4 of [3] the renormalization due to impurity scattering is also discussed.

5 Boltzmann Transport: Relaxation-Time Approximation

In the relaxation-time approximation of the Boltzmann transport[9] the expectation value of the charge current $\mathbf{J}^e$ and the heat current $\mathbf{J}^Q$ are given as[10]

$$J_x^e = \left\langle ev_x \right\rangle \equiv 2e \sum_{\mathbf{p}} v_x g_{\mathbf{p}}, \tag{37}$$

and

$$J_x^Q = \left\langle \xi_{\mathbf{p}} v_x \right\rangle \equiv 2 \sum_{\mathbf{p}} \xi_{\mathbf{p}} v_x g_{\mathbf{p}}, \tag{38}$$

where $g_{\mathbf{p}}$ is the deviation of the distribution function[11] from the equilibrium distribution function $f(\xi_{\mathbf{p}})$ defined by (14) and e is the charge of an electron ($e < 0$). Throughout the series of three notes the Zeeman splitting is neglected so that the spin degrees of freedom is accounted only by the degeneracy factor 2.

If a static electric field $\mathbf{E} = (E_x, 0, 0)$ is applied, the resulting deviation is

$$g_{\mathbf{p}} = eE_x v_x \tau \left(- \frac{\partial f(\xi_{\mathbf{p}})}{\partial \xi_{\mathbf{p}}} \right), \tag{39}$$

so that

$$J_x^e = 2e^2 E_x \sum_{\mathbf{p}} v_x^2 \tau \left(- \frac{\partial f(\xi_{\mathbf{p}})}{\partial \xi_{\mathbf{p}}} \right) \equiv \sigma_{xx} E_x. \tag{40}$$

In the same manner

$$J_x^Q = 2e E_x \sum_{\mathbf{p}} \xi_{\mathbf{p}} v_x^2 \tau \left(- \frac{\partial f(\xi_{\mathbf{p}})}{\partial \xi_{\mathbf{p}}} \right) \equiv \tilde{\alpha}_{xx} E_x. \tag{41}$$

If a static magnetic field $\mathbf{H}$ is applied additionally, the resulting $\mathbf{J}^e$ is

$$\mathbf{J}^e = 2e^2 \sum_{\mathbf{p}} \frac{v_x^2 \tau}{1 + (\omega_c \tau)^2} \left(- \frac{\partial f(\xi_{\mathbf{p}})}{\partial \xi_{\mathbf{p}}} \right) \left[\mathbf{E} - \frac{e\tau}{m} (\mathbf{H} \times \mathbf{E}) \right], \tag{42}$$

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[9]See, for example, Ziman: *Principles of the Theory of Solids*, 2nd edition (Cambridge Univ. Press, Cambridge, 1972) for the Boltzmann transport of electrons. The Boltzmann transport of Cooper pairs is discussed in §3.7 of [3].

[10] Here $\mathbf{J}^e = (J_x^e, J_y^e, J_z^e)$, $\mathbf{J}^Q = (J_x^Q, J_y^Q, J_z^Q)$ and $\mathbf{v_p} = (v_x, v_y, v_z)$.

[11] Since we describe the distribution function of quasi-particles by the Boltzmann equation, $\xi_{\mathbf{p}}$ and $v_{\mathbf{p}}$ represent the energy and the velocity of an quasi-particle with a renormalized mass m^* within the discussion of the Boltzmann transport. Namely, $\xi_{\mathbf{p}} = v_F \cdot (|\mathbf{p}| - p_F)$ and $\mathbf{v_p} = \mathbf{p}/m^*$ where p_F is the Fermi momentum and $v_F = p_F/m^*$. To keep the notation simple, we do not distinguish m^* and m in the following.

where $\omega_c = |e\mathbf{H}|/m$.

If a static temperature gradient ∇T is applied additionally, its effect is taken into account as

$$eE' = e\mathbf{E} - \xi_{\mathbf{p}}\frac{\nabla T}{T}, \tag{43}$$

so that[12]

$$\mathbf{J}^e = 2e^2 \sum_{\mathbf{p}} \frac{v_x^2 \tau}{1 + (\omega_c\tau)^2}\Big(-\frac{\partial f(\xi_{\mathbf{p}})}{\partial \xi_{\mathbf{p}}}\Big)\Big[\mathbf{E}' - \frac{e\tau}{m}(\mathbf{H} \times \mathbf{E}')\Big], \tag{44}$$

and

$$\mathbf{J}^Q = 2e \sum_{\mathbf{p}} \frac{v_x^2 \tau}{1 + (\omega_c\tau)^2}\, \xi_{\mathbf{p}}\Big(-\frac{\partial f(\xi_{\mathbf{p}})}{\partial \xi_{\mathbf{p}}}\Big)\Big[\mathbf{E}' - \frac{e\tau}{m}(\mathbf{H} \times \mathbf{E}')\Big]. \tag{45}$$

6 Boltzmann Transport: Exact Formula

The Boltzmann equation in linear order of $\mathbf{E}$ for an interacting electron system under static electromagnetic field is

$$e\mathbf{E} \cdot \mathbf{v_p}\frac{\partial f(\xi_{\mathbf{p}})}{\partial \xi_{\mathbf{p}}} + e\big(\mathbf{v_p} \times \mathbf{H}\big) \cdot \frac{\partial g_{\mathbf{p}}}{\partial \mathbf{p}} = C_{\mathbf{p}}, \tag{46}$$

where the collision term $C_{\mathbf{p}}$ is given as

$$C_{\mathbf{p}} = \sum_{\mathbf{p}'} \Big\{C_{\mathbf{p}\mathbf{p}'}g_{\mathbf{p}'} - C_{\mathbf{p}'\mathbf{p}}g_{\mathbf{p}}\Big\} \equiv -\sum_{\mathbf{p}'} \big(\tau_{\mathrm{tr}}^{-1}\big)_{\mathbf{p}\mathbf{p}'}g_{\mathbf{p}'}, \tag{47}$$

with

$$\big(\tau_{\mathrm{tr}}^{-1}\big)_{\mathbf{p}\mathbf{p}'} = \frac{1}{\tau_{\mathbf{p}}}\delta_{\mathbf{p},\mathbf{p}'} - C_{\mathbf{p}\mathbf{p}'}, \tag{48}$$

and

$$\frac{1}{\tau_{\mathbf{p}}} \equiv \sum_{\mathbf{p}'} C_{\mathbf{p}'\mathbf{p}}. \tag{49}$$

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[12] The treatment of the magnetic field in Reizer and Sergeev: Phys. Rev. B **61**, 7340 (2000) is not compatible with the Boltzmann transport. That in Sergeev, Reizer and Mitin: Phys. Rev. B **77**, 064501 (2008) is not compatible with the GL transport. In these works the gauge invariance is not satisfied in ordinary manner where the momentum is combined with the vector potential and the energy with the scalar potential. While the direction of the electron motion is affected, the value of the energy is not affected by the magnetic field. Some criticisms against these works have been made by Serbyn, Skvortsov and Varlamov: arXiv:1012.4316. However, Sergeev, Reizer and Mitin: arXiv:1101.4186 still claim their validity.

Here $C_{\mathbf{p'p}}$ is the transition rate from $\mathbf{p}$ to $\mathbf{p'}$ and has a symmetry $C_{\mathbf{p'p}} = C_{\mathbf{pp'}}$.

This linear Boltzmann equation is formally solved exactly[13] as follows. By introducing the matrix A

$$A_{\mathbf{pp'}} = \left(\tau_{\mathrm{tr}}^{-1}\right)_{\mathbf{pp'}} - e\left(\mathbf{v_p} \times \frac{\partial}{\partial \mathbf{p}}\right) \cdot \mathbf{H}\, \delta_{\mathbf{p},\mathbf{p'}}, \tag{50}$$

(46) is written in the form

$$\sum_{\mathbf{p'}} A_{\mathbf{pp'}} g_{\mathbf{p'}} = e\mathbf{E} \cdot \mathbf{v_p}\left(-\frac{\partial f(\xi_\mathbf{p})}{\partial \xi_\mathbf{p}}\right). \tag{51}$$

If we get the inverse matrix A^{-1}, the deviation of the distribution is known as

$$g_{\mathbf{p'}} = \sum_{\mathbf{p}} A_{\mathbf{p'p}}^{-1} e\mathbf{E} \cdot \mathbf{v_p}\left(-\frac{\partial f(\xi_\mathbf{p})}{\partial \xi_\mathbf{p}}\right). \tag{52}$$

Using the identity for operators P and Q

$$\left(P - Q\right)^{-1} = P^{-1} + P^{-1}QP^{-1} + P^{-1}QP^{-1}QP^{-1} + \cdots, \tag{53}$$

the inverse matrix is obtained as an expansion in terms of the magnetic field

$$A_{\mathbf{p'p}}^{-1} = \left(\tau_{\mathrm{tr}}\right)_{\mathbf{p'p}} + e\sum_{\mathbf{p_1}} \left(\tau_{\mathrm{tr}}\right)_{\mathbf{p'p_1}}\left[\left(\mathbf{v_{p_1}} \times \frac{\partial}{\partial \mathbf{p_1}}\right) \cdot \mathbf{H}\right]\left(\tau_{\mathrm{tr}}\right)_{\mathbf{p_1 p}}$$
$$+ e^2 \sum_{\mathbf{p_1}}\sum_{\mathbf{p_2}} \left(\tau_{\mathrm{tr}}\right)_{\mathbf{p'p_1}}\left[\left(\mathbf{v_{p_1}} \times \frac{\partial}{\partial \mathbf{p_1}}\right) \cdot \mathbf{H}\right]\left(\tau_{\mathrm{tr}}\right)_{\mathbf{p_1 p_2}}\left[\left(\mathbf{v_{p_2}} \times \frac{\partial}{\partial \mathbf{p_2}}\right) \cdot \mathbf{H}\right]\left(\tau_{\mathrm{tr}}\right)_{\mathbf{p_2 p}} + \cdots, \tag{54}$$

Employing (52) and (54) we can calculate

$$\mathbf{J}^e = 2e \sum_{\mathbf{p'}} \mathbf{v_{p'}} g_{\mathbf{p'}}, \tag{55}$$

and the conductivity tensor $\sigma^{\mu\nu}$ is introduced[14] as

$$J_\mu^e = \sum_{\nu} \sigma^{\mu\nu} E_\nu. \tag{56}$$

Thus the conductivity tensor is expressed as

$$\sigma^{\mu\nu} = 2e^2 \sum_{\mathbf{p'}}\sum_{\mathbf{p}} v_{\mathbf{p'}}^\mu A_{\mathbf{p'p}}^{-1} v_{\mathbf{p}}^\nu\left(-\frac{\partial f(\xi_\mathbf{p})}{\partial \xi_\mathbf{p}}\right), \tag{57}$$

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[13] See, for example, Kotliar, Sengupta and Varma: Phys. Rev. B **53**, 3573 (1996).
[14] Here $\mathbf{J}^e = (J_x^e, J_y^e, J_z^e)$, $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{v_p} = (v_\mathbf{p}^x, v_\mathbf{p}^y, v_\mathbf{p}^z)$ with $\mu, \nu = x, y, z$.

and this expression is microscopically derived on the basis of the Fermi-liquid theory.[15] The conductivity in the absence of magnetic field is obtained as

$$\sigma^{xx} = 2e^2 \sum_{\mathbf{p}'} \sum_{\mathbf{p}} v^x_{\mathbf{p}'} (\tau_{\text{tr}})_{\mathbf{p}'\mathbf{p}} v^x_{\mathbf{p}} \Big(-\frac{\partial f(\xi_{\mathbf{p}})}{\partial \xi_{\mathbf{p}}} \Big). \tag{58}$$

7 Charge and Heat Currents

The charge current of electrons, created by $\psi^\dagger_\sigma(\mathbf{x})$ and annihilated by $\psi_\sigma(\mathbf{x})$, is

$$\mathbf{j}^e(\mathbf{x}) = \frac{e}{2mi} \sum_\sigma \Big[\psi^\dagger_\sigma(\mathbf{x}) \big(\nabla \psi_\sigma(\mathbf{x}) \big) - \big(\nabla \psi^\dagger_\sigma(\mathbf{x}) \big) \psi_\sigma(\mathbf{x}) \Big], \tag{59}$$

and satisfies the conservation law

$$\dot{\rho}(\mathbf{x}) + \nabla \cdot \mathbf{j}^e(\mathbf{x}) = 0, \tag{60}$$

where

$$\rho(\mathbf{x}) = e \sum_\sigma \psi^\dagger_\sigma(\mathbf{x}) \psi_\sigma(\mathbf{x}) \equiv j^e_0(\mathbf{x}), \tag{61}$$

is the charge density. Using the four-divergence (60) is written as

$$\sum_{\mu=0}^{3} \frac{\partial}{\partial x_\mu} j^e_\mu(x) = 0, \tag{62}$$

where $x = (t, \mathbf{x}) = (x_0, x_1, x_2, x_3)$ with $\mu = 0, 1, 2, 3$.

The energy current[16] of electrons is

$$\mathbf{u}(\mathbf{x}) = -\frac{1}{2m} \sum_\sigma \Big[\big(\nabla \psi^\dagger_\sigma(\mathbf{x}) \big) \dot{\psi}_\sigma(\mathbf{x}) + \dot{\psi}^\dagger_\sigma(\mathbf{x}) \big(\nabla \psi_\sigma(\mathbf{x}) \big) \Big], \tag{63}$$

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[15] The microscopic derivation of the conductivity is done by Éliashberg: Sov. Phys. JETP **14**, 886 (1962). This work is extended to the case of the Hall conductivity by Kohno and Yamada: Prog. Theor. Phys. **80**, 623 (1988) and to the magneto-conductivity by Kontani: Phys. Rev. B **64**, 054413 (2001).

[16] Via the energy-momentum tensor the energy current is given as

$$u_\mu = \sum_\sigma \Big(\frac{\partial \mathcal{L}}{\partial(\partial \psi_\sigma / \partial x_\mu)} \dot{\psi}_\sigma + \dot{\psi}^\dagger_\sigma \frac{\partial \mathcal{L}}{\partial(\partial \psi^\dagger_\sigma / \partial x_\mu)} \Big),$$

where $\mu = 1, 2, 3$. The Lagrangian density $\mathcal{L}$ can be replaced by its kinetic part

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2m} \sum_\sigma \nabla \psi^\dagger_\sigma \cdot \nabla \psi_\sigma,$$

because its interaction part $\mathcal{L}_{\text{int}}$ does not contain the derivative of $\psi^\dagger_\sigma$ and ψ_σ. See, for example, Langer: Phys. Rev. **128**, 110 (1962) or §10.3 in [2].

and satisfies the conservation law

$$\dot{h}(\mathbf{x}) + \nabla \cdot \mathbf{u}(\mathbf{x}) = 0, \tag{64}$$

where $h(\mathbf{x})$ is the Hamiltonian density.

According to the thermodynamic relation $dQ = dU - \mu N$, the heat current is

$$\mathbf{j}^Q(\mathbf{x}) = \mathbf{u}(\mathbf{x}) - \frac{\mu}{e}\mathbf{j}^e(\mathbf{x}), \tag{65}$$

and satisfies the conservation law

$$\sum_{\mu=0}^{3} \frac{\partial}{\partial x_\mu} j_\mu^Q(x) = 0, \tag{66}$$

where

$$j_0^Q(x) = h(\mathbf{x}) - \frac{\mu}{e}\rho(\mathbf{x}). \tag{67}$$

8 Kubo Formula: Linear Response to Electric Field

We observe the expectation value of the charge current $\mathbf{J}^e$ caused by the external electric field $\mathbf{E}$ as

$$J_\mu^e(\mathbf{k}, \omega) = \sum_\nu \sigma_{\mu\nu}(\mathbf{k}, \omega) E_\nu(\mathbf{k}, \omega). \tag{68}$$

The conductivity tensor σ is given by the Kubo formula[17] (linear response theory)

$$\Phi_{\mu\nu}^e(\mathbf{k}, i\omega_\lambda) = \int_0^\beta d\tau \, e^{i\omega_\lambda \tau} \big\langle T_\tau \big\{ j_\mu^e(\mathbf{k}; \tau) \, j_\nu^e(\mathbf{k}{=}0) \big\} \big\rangle, \tag{69}$$

and

$$\sigma_{\mu\nu}(\mathbf{k}, \omega) = \frac{1}{i\omega} \big[\Phi_{\mu\nu}^e(\mathbf{k}, \omega + i\delta) - \Phi_{\mu\nu}^e(\mathbf{k}, i\delta) \big]. \tag{70}$$

Here

$$j_\mu^e(\mathbf{k}; \tau) = e^{K\tau} \, j_\mu^e(\mathbf{k}) \, e^{-K\tau}, \tag{71}$$

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[17] I follow Fukuyama, Ebisawa and Wada: Prog. Theor. Phys. **42**, 494 (1969) concerning the use of the Kubo formula. The sign of $\Phi_{\mu\nu}^e$ is opposite to $Q_{\mu\nu}$ in [3] and [AGD] and to $K_{\mu\nu}$ in [4] and [FW].

[AGD] $\equiv$ Abrikosov, Gorkov and Dzyaloshinski: *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1975).

[FW] $\equiv$ Fetter and Walecka: *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971).

and

$$\mathbf{j}^e(\mathbf{k}) = \int d^3x \, e^{-i\mathbf{k}\cdot\mathbf{x}} \, \mathbf{j}^e(\mathbf{x}). \tag{72}$$

The Fourier transform of (59) results in

$$\mathbf{j}^e(\mathbf{k}) = e \sum_{\mathbf{p}} \left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}+\mathbf{k}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}+\mathbf{k}}\right)$$

$$= e \sum_{\mathbf{p}} \frac{1}{2}\left(\mathbf{v}_{\mathbf{p}} + \mathbf{v}_{\mathbf{p}+\mathbf{k}}\right)\left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}+\mathbf{k}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}+\mathbf{k}}\right), \tag{73}$$

and in the limit of $\mathbf{k} \to 0$

$$\mathbf{j}^e(\mathbf{k}=0) = e \sum_{\mathbf{p}} \mathbf{v}_{\mathbf{p}}\left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}}\right) = e \sum_{\mathbf{p}} \sum_{\sigma} \mathbf{v}_{\mathbf{p}} c_{\mathbf{p}\sigma}^\dagger c_{\mathbf{p}\sigma}, \tag{74}$$

where $c_{\mathbf{p}\sigma}^\dagger$ represents the creation operator of an electron with momentum $\mathbf{p}$ and spin σ and $c_{\mathbf{p}\sigma}$ represents the annihilation operator ($\sigma = \uparrow, \downarrow$) introduced by

$$\psi_\sigma^\dagger(\mathbf{x}) = \sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} c_{\mathbf{p}\sigma}^\dagger, \qquad \psi_\sigma(\mathbf{x}) = \sum_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} c_{\mathbf{p}\sigma}. \tag{75}$$

In the presence of the temperature gradient both $\mathbf{E}$ and ∇T are the causes of the observed charge current $\mathbf{J}^e$ and heat current $\mathbf{J}^Q$ as [5]

$$\begin{pmatrix} \mathbf{J}^e \\ \mathbf{J}^Q \end{pmatrix} \begin{pmatrix} \sigma & \alpha \\ \tilde{\alpha} & \kappa \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ -\nabla T \end{pmatrix}, \tag{76}$$

where κ is the thermal conductivity tensor and α and $\tilde{\alpha}$ are thermo-electric tensors. Here $\tilde{\alpha}$ is obtained by the Kubo formula as follows and α is obtained via the Onsager relation[18] as

$$\alpha = \frac{1}{T}\tilde{\alpha}. \tag{77}$$

The expectation value of the heat current $\mathbf{J}^Q$ caused by the external electric field $\mathbf{E}$ is

$$J_\mu^Q(\mathbf{k}, \omega) = \sum_\nu \tilde{\alpha}_{\mu\nu}(\mathbf{k}, \omega) E_\nu(\mathbf{k}, \omega), \tag{78}$$

and the thermo-electric tensor $\tilde{\alpha}$ is given by the Kubo formula

$$\Phi_{\mu\nu}^Q(\mathbf{k}, i\omega_\lambda) = \int_0^\beta d\tau \, e^{i\omega_\lambda \tau} \langle T_\tau \{ j_\mu^Q(\mathbf{k}; \tau) \, j_\nu^e(\mathbf{k}=0) \} \rangle, \tag{79}$$

---

[18] We have regarded $\mathbf{E}$ and $-\nabla T/T$ as external forces and assumed the Onsager relation between the responses to these forces.

and

$$\tilde{\alpha}_{\mu\nu}(\mathbf{k}, \omega) = \frac{1}{i\omega} \left[\Phi_{\mu\nu}^Q(\mathbf{k}, \omega + i\delta) - \Phi_{\mu\nu}^Q(\mathbf{k}, i\delta) \right]. \tag{80}$$

Here

$$j_\mu^Q(\mathbf{k}; \tau) = e^{K\tau} \, j_\mu^Q(\mathbf{k}) \, e^{-K\tau}, \tag{81}$$

and

$$\mathbf{j}^Q(\mathbf{k}) = \int d^3x \, e^{-i\mathbf{k}\cdot\mathbf{x}} \, \mathbf{j}^Q(\mathbf{x}). \tag{82}$$

9 Jonson-Mahan Formula: Relation between Response Functions

Let us find the relation between the energy current

$$u_x(t) = -\frac{1}{2m} \sum_\sigma \left(\dot{\psi}_\sigma^\dagger(t) \frac{\partial \psi_\sigma(t)}{\partial x} + \frac{\partial \psi_\sigma^\dagger(t)}{\partial x} \dot{\psi}_\sigma(t) \right), \tag{83}$$

and the charge current

$$j_x^e(t) = \frac{1}{2mi} \sum_\sigma \left(\psi_\sigma^\dagger(t) \frac{\partial \psi_\sigma(t)}{\partial x} - \frac{\partial \psi_\sigma^\dagger(t)}{\partial x} \psi_\sigma(t) \right). \tag{84}$$

Introducing a two-time function

$$J_\sigma(t, t') = \left(\psi_\sigma^\dagger(t) \frac{\partial \psi_\sigma(t')}{\partial x} - \frac{\partial \psi_\sigma^\dagger(t)}{\partial x} \psi_\sigma(t') \right), \tag{85}$$

the energy current is written as[19]

$$u_x = -\lim_{t' \to t} \frac{1}{2} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right) \frac{1}{2m} \sum_\sigma J_\sigma(t, t'). \tag{86}$$

Since

$$-i\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}, \tag{87}$$

---

[19] To determine (63) we have chosen one of the equivalent forms of the density $\mathcal{L}_{\text{kin}}$ resulting in the same value after integration $L = \int d^3x \mathcal{L}_{\text{kin}}$. Here the equivalent forms of the current density is introduced via the integration by parts

$$\int d^3x \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right) J_\sigma(t, t') = 2 \int d^3x \left(\dot{\psi}_\sigma^\dagger(t) \frac{\partial \psi_\sigma(t')}{\partial x} + \frac{\partial \psi_\sigma^\dagger(t)}{\partial x} \dot{\psi}_\sigma(t') \right),$$

where we have neglected the contribution of the surface.

we obtain

$$j_x^Q = \lim_{\tau' \to \tau} \frac{1}{2}\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\tau'}\right)\frac{1}{2mi}\sum_\sigma J_\sigma(\tau, \tau'). \tag{88}$$

Thus Φ_{xx}^e is expressed as

$$\Phi_{xx}^e(\mathbf{k}=0, i\omega_\lambda) = e\lim_{\tau' \to \tau}\int_0^\beta d\tau\, e^{i\omega_\lambda\tau}F(\tau, \tau'), \tag{89}$$

where we have introduced a function

$$F(\tau, \tau') \equiv \sum_{\mathbf{p}}\sum_\sigma v_x\langle c_{\mathbf{p}\sigma}^\dagger(\tau)c_{\mathbf{p}\sigma}(\tau')j_x^e(\mathbf{k}=0)\rangle, \tag{90}$$

via the Fourier transform of (85), with $v_x = p_x/m$. Here we have assumed that $\beta > \tau > \tau' > 0$ and dropped T_τ from (69). Introducing a function

$$S(\tau, \tau') = \frac{1}{2}\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\tau'}\right)F(\tau, \tau'), \tag{91}$$

Φ_{xx}^Q is expressed as

$$\Phi_{xx}^Q(\mathbf{k}=0, i\omega_\lambda) = \lim_{\tau' \to \tau}\int_0^\beta d\tau\, e^{i\omega_\lambda\tau}S(\tau, \tau'). \tag{92}$$

Employing the Fourier transform[20]

$$F(\tau, \tau') = \frac{1}{\beta^2}\sum_n\sum_{n'} F(i\varepsilon_n, i\varepsilon_{n'})e^{i(\varepsilon_n\tau - \varepsilon_{n'}\tau')}, \tag{93}$$

---

[20] Neglecting the vertex correction

$$F(\tau, \tau') = e\sum_{\mathbf{p}\mathbf{p}'}\sum_{\sigma\sigma'} v_x\langle c_{\mathbf{p}\sigma}^\dagger(\tau)c_{\mathbf{p}\sigma}(\tau')c_{\mathbf{p}'\sigma'}^\dagger(0)c_{\mathbf{p}'\sigma'}(0)\rangle v_x',$$

is factorized as

$$F(\tau, \tau') = e\sum_{\mathbf{p}}\sum_\sigma v_x^2\langle c_{\mathbf{p}\sigma}(\tau')c_{\mathbf{p}\sigma}^\dagger(0)\rangle\langle c_{\mathbf{p}\sigma}^\dagger(\tau)c_{\mathbf{p}\sigma}(0)\rangle.$$

Since

$$\langle c_{\mathbf{p}\sigma}(\tau')c_{\mathbf{p}\sigma}^\dagger(0)\rangle = -G(\mathbf{p}, \tau'), \qquad \langle c_{\mathbf{p}\sigma}^\dagger(\tau)c_{\mathbf{p}\sigma}(0)\rangle = G(\mathbf{p}, -\tau),$$

for $\tau, \tau' > 0$,

$$F(\tau, \tau') = -e\sum_{\mathbf{p}}\sum_\sigma v_x^2 G(\mathbf{p}, \tau')G(\mathbf{p}, -\tau).$$

Via the Fourier transform of the propagator

$$G(\mathbf{p}, \tau') = \frac{1}{\beta}\sum_{n'} G(\mathbf{p}, i\varepsilon_{n'})e^{-i\varepsilon_{n'}\tau'}, \qquad G(\mathbf{p}, -\tau) = \frac{1}{\beta}\sum_n G(\mathbf{p}, i\varepsilon_n)e^{i\varepsilon_n\tau},$$

$F(\tau, \tau')$ in this approximation is shown to have the form of (93).

14

with ε_n and $\varepsilon_{n'}$ being the fermionic thermal frequency, Φ^e_{xx} is expressed as

$$\Phi^e_{xx}(\mathbf{k}=0, i\omega_\lambda) = \frac{e}{\beta} \sum_n F(i\varepsilon_n, i\varepsilon_n + i\omega_\lambda). \tag{94}$$

Since

$$S(\tau, \tau') = \frac{1}{2\beta^2} \sum_n \sum_{n'} (i\varepsilon_n + i\varepsilon_{n'}) F(i\varepsilon_n, i\varepsilon_{n'}) e^{i(\varepsilon_n \tau - \varepsilon_{n'}\tau')}, \tag{95}$$

we obtain[21]

$$\Phi^Q_{xx}(\mathbf{k}=0, i\omega_\lambda) = \frac{1}{\beta} \sum_n \left(i\varepsilon_n + \frac{i\omega_\lambda}{2} \right) F(i\varepsilon_n, i\varepsilon_n + i\omega_\lambda). \tag{96}$$

Thus Φ^e_{xx} and Φ^Q_{xx} are related[22] by (94) and (96) for electrons.

If we regard the Cooper pair as the carrier[23] of charge and heat, we can introduce phenomenologically the current operators as

$$\mathbf{j}^e(\mathbf{x}) = \frac{e^*}{2m^*i} \left[\Psi^\dagger(\mathbf{x}) \Big(\nabla \Psi(\mathbf{x}) \Big) - \Big(\nabla \Psi^\dagger(\mathbf{x}) \Big) \Psi(\mathbf{x}) \right], \tag{97}$$

and

$$\mathbf{j}^Q(\mathbf{x}) = -\frac{1}{2m^*} \left[\Big(\nabla \Psi^\dagger(\mathbf{x}) \Big) \dot{\Psi}(\mathbf{x}) + \dot{\Psi}^\dagger(\mathbf{x}) \Big(\nabla \Psi(\mathbf{x}) \Big) \right], \tag{98}$$

where

$$\Psi^\dagger(\mathbf{x}) = \sum_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{x}} P^\dagger_{\mathbf{q}}, \qquad \Psi(\mathbf{x}) = \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}} P_{\mathbf{q}}. \tag{99}$$

These are the GL expressions of the currents [5]. The effective charge and mass of Cooper pairs are given as

$$e^* = 2e, \qquad m^* = 2m. \tag{100}$$

If we consider the vertex correction, we obtain

$$F(\tau, \tau') = - \sum_{\mathbf{p}} \sum_{\sigma} v_x G(\mathbf{p}, \tau') G(\mathbf{p}, -\tau) \tilde{j}^e_x,$$

where $\tilde{j}^e_x$ is the renormalized current vertex so that (93) holds in general.

[21] We have employed

$$\int_0^\beta d\tau\, e^{i\omega_\lambda \tau} e^{i(\varepsilon_n - \varepsilon_{n'})\tau} = \beta \delta_{\varepsilon_n + \omega_\lambda, \varepsilon_{n'}}.$$

[22] This relation is obtained by Jonson and Mahan: Phys. Rev. B **42**, 9350 (1990) and also discussed by Mahan: Solid State Physics **51**, 81 (1998).

[23] Since $\mu = 0$ for Cooper pairs, the heat and energy currents are identified, $\mathbf{j}^Q(\mathbf{x}) = \mathbf{u}(\mathbf{x})$, here.

Employing the Cooper-pair currents (97) and (98) instead of the electron currents (59) and (63), we obtain

$$\Phi_{xx}^e(\mathbf{k}{=}0, i\omega_\lambda) = \frac{e^*}{\beta}\sum_m \tilde{F}(i\omega_m, i\omega_m + i\omega_\lambda), \qquad (101)$$

and

$$\Phi_{xx}^Q(\mathbf{k}{=}0, i\omega_\lambda) = \frac{1}{\beta}\sum_m \left(i\omega_m + \frac{i\omega_\lambda}{2}\right)\tilde{F}(i\omega_m, i\omega_m + i\omega_\lambda), \qquad (102)$$

where

$$\tilde{F}(\tau, \tau') \equiv \sum_{\mathbf{q}} v_x^* \big\langle P_{\mathbf{q}}^\dagger(\tau) P_{\mathbf{q}}(\tau') j_x^e(\mathbf{k}{=}0)\big\rangle, \qquad (103)$$

with ω_m being the bosonic thermal frequency and $v_x^* = q_x/m^*$. Thus Φ_{xx}^e and Φ_{xx}^Q are related by (101) and (102) for Cooper pairs.

10   Ward Identities: Exact Results
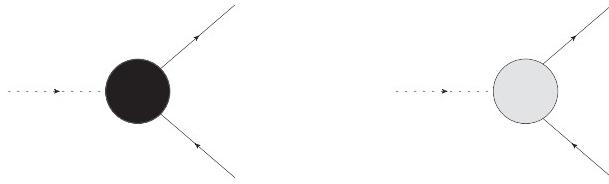
Figure 2: Current vertices for quasi-particles: The left black one is the charge current vertex. The right gray one is the heat current vertex. The broken line with arrow represents the coupling to the external field.

First we discuss the derivation [4] of the Ward identity for charge current vertex at zero temperature. We introduce the three-point function Λ_μ^e, ($\mu = 1, 2, 3, 0$), defined by

$$\Lambda_\mu^e(x, y, z) = \big\langle T\big\{j_\mu^e(z)\psi_\uparrow(x)\psi_\uparrow^\dagger(y)\big\}\big\rangle, \qquad (104)$$

where $\langle A\rangle$ represents the expectation value of A in the ground state, T is the time-ordering operator and $z = (\mathbf{z}, z_0)$ with coordinate vector $\mathbf{z} = (z_1, z_2, z_3)$ and time z_0. Here $\psi_\uparrow(x)$ and $\psi_\uparrow^\dagger(y)$ are annihilation and creation operators

of ↑-spin electron. The charge current j_μ^e of electrons obeys the charge-conservation law

$$\sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu} j_\mu^e(z) = 0, \tag{105}$$

where

$$j_0^e(z) = e\psi_\uparrow^\dagger(z)\psi_\uparrow(z) + e\psi_\downarrow^\dagger(z)\psi_\downarrow(z). \tag{106}$$

The time ordering of three operators results in the summation of 3! terms as

$$\begin{aligned}
\Lambda_\mu^e(x,y,z) = &\left\langle j_\mu^e(z)\psi_\uparrow(x)\psi_\uparrow^\dagger(y)\right\rangle\theta(z_0-x_0)\theta(x_0-y_0) \\
&- \left\langle j_\mu^e(z)\psi_\uparrow^\dagger(y)\psi_\uparrow(x)\right\rangle\theta(z_0-y_0)\theta(y_0-x_0) \\
&+ \left\langle \psi_\uparrow(x)j_\mu^e(z)\psi_\uparrow^\dagger(y)\right\rangle\theta(x_0-z_0)\theta(z_0-y_0) \\
&- \left\langle \psi_\uparrow^\dagger(y)j_\mu^e(z)\psi_\uparrow(x)\right\rangle\theta(y_0-z_0)\theta(z_0-x_0) \\
&+ \left\langle \psi_\uparrow(x)\psi_\uparrow^\dagger(y)j_\mu^e(z)\right\rangle\theta(x_0-y_0)\theta(y_0-z_0) \\
&- \left\langle \psi_\uparrow^\dagger(y)\psi_\uparrow(x)j_\mu^e(z)\right\rangle\theta(y_0-x_0)\theta(x_0-z_0). \tag{107}
\end{aligned}$$

Thus the time-derivative of Λ_μ^e results in

$$\begin{aligned}
\frac{\partial}{\partial z_0}\Lambda_0^e(x,y,z) = &\,\delta(z_0-x_0)\Big(\theta(x_0-y_0)\left\langle \left[j_0^e(z),\psi_\uparrow(x)\right]\psi_\uparrow^\dagger(y)\right\rangle \\
&\qquad\qquad -\theta(y_0-x_0)\left\langle\psi_\uparrow^\dagger(y)\left[j_0^e(z),\psi_\uparrow(x)\right]\right\rangle\Big) \\
&+\delta(z_0-y_0)\Big(\theta(x_0-y_0)\left\langle\psi_\uparrow(x)\left[j_0^e(z),\psi_\uparrow^\dagger(y)\right]\right\rangle \\
&\qquad\qquad -\theta(y_0-x_0)\left\langle\left[j_0^e(z),\psi_\uparrow^\dagger(y)\right]\psi_\uparrow(x)\right\rangle\Big) \\
&\qquad\qquad +\left\langle T\left\{\frac{\partial j_0^e(z)}{\partial z_0}\psi_\uparrow(x)\psi_\uparrow^\dagger(y)\right\}\right\rangle. \tag{108}
\end{aligned}$$

Adding the divergence in terms of three coordinate variables and using again the time ordering operator T we obtain the four-divergence of Λ_μ^e as

$$\begin{aligned}
\sum_{\mu=0}^{3}\frac{\partial}{\partial z_\mu}\Lambda_\mu^e(x,y,z) = &\left\langle T\left\{\left[j_0^e(z),\psi_\uparrow(x)\right]\psi_\uparrow^\dagger(y)\right\}\right\rangle\delta(z_0-x_0) \\
&+\left\langle T\left\{\psi_\uparrow(x)\left[j_0^e(z),\psi_\uparrow^\dagger(y)\right]\right\}\right\rangle\delta(z_0-y_0) \\
&+\left\langle T\left\{\sum_{\mu=0}^{3}\frac{\partial j_\mu^e(z)}{\partial z_\mu}\psi_\uparrow(x)\psi_\uparrow^\dagger(y)\right\}\right\rangle. \tag{109}
\end{aligned}$$

The last term on the right-hand side vanishes due to the charge-conservation law (105). Only equal space-time commutation relations are non-vanishing,

$$\left[j_0^e(z), \psi_\uparrow^\dagger(y)\right]\delta(z_0 - y_0) = e\psi_\uparrow^\dagger(y)\delta^4(z - y), \tag{110}$$

and

$$\left[j_0^e(z), \psi_\uparrow(x)\right]\delta(z_0 - x_0) = -e\psi_\uparrow(x)\delta^4(z - x), \tag{111}$$

so that the non-vanishing contribution becomes

$$\sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu}\Lambda_\mu^e(x, y, z) = -e\langle T\{\psi_\uparrow(x)\psi_\uparrow^\dagger(y)\}\rangle\delta^4(z - x)$$
$$+ e\langle T\{\psi_\uparrow(x)\psi_\uparrow^\dagger(y)\}\rangle\delta^4(z - y). \tag{112}$$

Introducing the electron propagator $G(x, y)$ as

$$G(x, y) = -i\langle T\{\psi_\uparrow(x)\psi_\uparrow^\dagger(y)\}\rangle, \tag{113}$$

this relation is written into

$$\sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu}\Lambda_\mu^e(x, y, z) = -ieG(x, y)\delta^4(z - x) + ieG(x, y)\delta^4(z - y). \tag{114}$$

Assuming the translational invariance we set $y = 0$ and introduce the Fourier transform as

$$\Lambda_\mu^e(p, k) = \int d^4x e^{-ipx} \int d^4z e^{-ikz} \langle T\{j_\mu^e(z)\psi_\uparrow(x)\psi_\uparrow^\dagger(0)\}\rangle, \tag{115}$$

where the four-momentum is defined as $p = (\mathbf{p}, p_0)$ and $k = (\mathbf{k}, k_0)$. The left-hand side of (114) is evaluated as

$$\sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu}\Lambda_\mu^e(x, 0, z) = \int \frac{d^4p}{(2\pi)^4}e^{ipx} \int \frac{d^4k}{(2\pi)^4}e^{ikz} \sum_{\mu=0}^{3} ik_\mu\Lambda_\mu^e(p, k), \tag{116}$$

and the right-hand side is transformed as

$$\int d^4x e^{-ipx} \int d^4z e^{-ikz}\left(-G(x,0)\delta^4(z - x) + G(x,0)\delta^4(z)\right)$$
$$= -G(p + k) + G(p), \tag{117}$$

where

$$G(p) = \int d^4x e^{-ipx}G(x, 0). \tag{118}$$

18

Therefore we obtain

$$\sum_{\mu=0}^{3} k_\mu \Lambda_\mu^e(p,k) = eG(p) - eG(p+k). \tag{119}$$

It should be noted that the factor e represents the charge carried by an electron and is automatically taken into account by the commutation relation. The vertex function Γ_μ^e is introduced as

$$\Lambda_\mu^e(p,k) = iG(p) \cdot \Gamma_\mu^e(p,k) \cdot iG(p+k), \tag{120}$$

in accordance with the definiton of the Green function (113). Then the Ward identity for the charge current vertex is obtained as

$$\sum_{\mu=0}^{3} k_\mu \Gamma_\mu^e(p,k) = eG(p)^{-1} - eG(p+k)^{-1}. \tag{121}$$

Since the Fourier transform is introduced as

$$px = p_1 x_1 + p_2 x_2 + p_3 x_3 - \epsilon t, \tag{122}$$

where ϵ is the energy and t is the time, $x_0 = t$ and $p_0 = -\epsilon$ (and in the same manner $k_0 = -\omega$ with ω being the energy of the external field) in the zero-temperatute formalism.

In the finite-temperature formalism we employ the time-ordering operator T_τ and consider the three-point function

$$\Lambda_\mu^e(x,y,z) = \left\langle T_\tau \left\{ j_\mu^e(z) \psi_\uparrow(x) \psi_\uparrow^\dagger(y) \right\} \right\rangle, \tag{123}$$

where the real time z_0 and the imaginary time τ_z is related by $\tau_z = iz_0$ and $\langle A \rangle$ represents the expectation value of A by grand canonical ensemble. Taking the charge-conservation law (105) into account we obtain[24]

$$-i \sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu} \Lambda_\mu^e(x,y,z) = \left\langle T_\tau \left\{ \left[j_0^e(z), \psi_\uparrow(x) \right] \psi_\uparrow^\dagger(y) \right\} \right\rangle \delta(\tau_z - \tau_x)$$

$$+ \left\langle T_\tau \left\{ \psi_\uparrow(x) \left[j_0^e(z), \psi_\uparrow^\dagger(y) \right] \right\} \right\rangle \delta(\tau_z - \tau_y). \tag{124}$$

---

[24] The derivative $\partial \Lambda_0^e(x,y,z)/\partial \tau_z$ results in the same form as (108) where, since $\partial/\partial \tau = -i\partial/\partial t$, the charge-conservation law becomes

$$\frac{\partial \rho(\mathbf{x})}{\partial \tau} - i\nabla \cdot \mathbf{j}^e(\mathbf{x}) = 0.$$

Here we have used the relation

$$\frac{\partial}{\partial \tau_z} = -i \frac{\partial}{\partial z_0}. \qquad (125)$$

The finite-temperature vertex function Γ_μ^e is introduced as

$$\Lambda_\mu^e(p,k) = \big[- G(p) \big] \cdot \Gamma_\mu^e(p,k) \cdot \big[- G(p+k) \big], \qquad (126)$$

in accordance with the definiton of the thermal Green function

$$G(x,y) = -\big\langle T_\tau \big\{ \psi_\uparrow(x) \psi_\uparrow^\dagger(y) \big\} \big\rangle. \qquad (127)$$

Then the resulting Ward identity is the same form as (121) in the case of zero temperature. For the finite-temperature Ward identity the zeroth component of the four-momentum is $p_0 = -i\varepsilon_n$ with fermionic thermal frequency ε_n and $k_0 = -i\omega_\lambda$ with bosonic thermal frequency ω_λ.
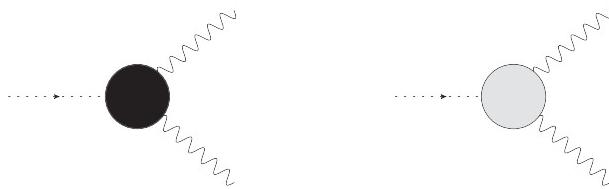
Figure 3: Current vertices for Cooper pairs.

The extension[25] of this Ward identity to the case of Cooper pairs is straightforward. In this note we only consider the local Cooper pair with s-wave symmetry in consistent with the local attractive interaction (5). Replacing $\psi_\uparrow(x)$ by $\Psi(x) = \psi_\downarrow(x)\psi_\uparrow(x)$ and $\psi_\uparrow^\dagger(y)$ by $\Psi^\dagger(y) = \psi_\uparrow^\dagger(y)\psi_\downarrow^\dagger(y)$ in (104) we consider the three-point function M_μ^e as

$$M_\mu^e(x,y,z) = \big\langle T \big\{ j_\mu^e(z) \Psi(x) \Psi^\dagger(y) \big\} \big\rangle, \qquad (128)$$

where $\Psi(x)$ and $\Psi^\dagger(y)$ are annihilation and creation operators of a Cooper pair which has a bosonic character. Using the commutation relation

$$\big[j_0^e(z), \Psi^\dagger(y) \big] \delta(z_0 - y_0) = 2e \Psi^\dagger(y) \delta^4(z-y), \qquad (129)$$

---

[25] I have derived the Ward identities for Cooper pairs with s-wave symmetry in arXiv:1108.0815. The Ward identities for anisotropic Cooper pairs have been discussed in arXiv:1108.5272. The Ward identities for Cooper pairs have been discussed in comparison with those for CDW and SDW in arXiv:1109.1404.

[Nar] $\equiv$ arXiv:1108.0815, arXiv:1108.5272, arXiv:1109.1404.

and

$$\big[j_0^e(z), \Psi(x)\big]\delta(z_0 - x_0) = -2e\Psi(x)\delta^4(z - x), \tag{130}$$

the four-divergence of M_μ^e is expressed as

$$\sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu} M_\mu^e(x, y, z) = -2e\big\langle T\big\{\Psi(x)\Psi^\dagger(y)\big\}\big\rangle \delta^4(z - x)$$
$$+ 2e\big\langle T\big\{\Psi(x)\Psi^\dagger(y)\big\}\big\rangle \delta^4(z - y), \tag{131}$$

by repeating the same calculations as those for deriving (112). Here the difference between fermion and boson is handled solely by the time-ordering operator T so that the expression of the divergence is common to fermion and boson. Introducing the Cooper-pair propagator $D(x, y)$ as

$$D(x, y) = -i\big\langle T\big\{\Psi(x)\Psi^\dagger(y)\big\}\big\rangle, \tag{132}$$

we obtain the Ward identity

$$\sum_{\mu=0}^{3} k_\mu \Delta_\mu^e(q, k) = 2eD(q)^{-1} - 2eD(q + k)^{-1}, \tag{133}$$

for Cooper pairs where Δ_μ^e is the counterpart of Γ_μ^e and $D(q)$ is the Fourier transform of $D(x, 0)$ with four-momentum $q = (\mathbf{q}, q_0)$. It should be noted that the factor $2e$ represents the charge carried by a Cooper pair and is automatically taken into account by the commutation relation.

Although the above derivation for Cooper pairs is formulated at zero temperature, (133) also holds at finite temperature with q_0 being a bosonic thermal frequency ($q_0 = -i\omega_m$). We are mainly interested in the normal metallic phase ($T > T_c$), the Cooper-pair propagator is a fluctuation propagator in this case.

Next we discuss the derivation of the Ward identity for heat current vertex.[26] Here we only consider the local interaction.[27] We introduce the three-point function Λ_μ^Q defined by

$$\Lambda_\mu^Q(x, y, z) = \big\langle T_\tau\big\{j_\mu^Q(z)\psi_\uparrow(x)\psi_\uparrow^\dagger(y)\big\}\big\rangle. \tag{134}$$

The heat current j_μ^Q obeys the conservation law

$$\sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu} j_\mu^Q(z) = 0, \tag{135}$$

---

[26] See [Ono] ≡ Ono: Prog. Theor. Phys. **46**, 757 (1971).

[27] The derivation becomes very simple in the case of the local interaction [Kon]. See [Ono], [Kon], [Nar] for the derivation in the case of non-local interaction.

[Kon] ≡ Kontani: Phys. Rev. B **67**, 014408 (2003).

where $j_0^Q(z)$ is given by (67) and is the density of (1). The four-divergence of Λ_μ^Q at finite temperature becomes

$$-i \sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu} \Lambda_\mu^Q(x,y,z) = \langle T_\tau \{ [j_0^Q(z), \psi_\uparrow(x)] \psi_\uparrow^\dagger(y) \} \rangle \delta(\tau_z - \tau_x)$$
$$+ \langle T_\tau \{ \psi_\uparrow(x) [j_0^Q(z), \psi_\uparrow^\dagger(y)] \} \rangle \delta(\tau_z - \tau_y), \qquad (136)$$

as (124). Since the interaction among electrons is local, $[j_0^Q(x), \psi_\uparrow(x)] = [K, \psi_\uparrow(x)]$ and $[j_0^Q(y), \psi_\uparrow^\dagger(y)] = [K, \psi_\uparrow^\dagger(y)]$ so that using (8)

$$[j_0^Q(x), \psi_\uparrow(x)] = -i \frac{\partial}{\partial x_0} \psi_\uparrow(x), \quad [j_0^Q(y), \psi_\uparrow^\dagger(y)] = -i \frac{\partial}{\partial y_0} \psi_\uparrow^\dagger(y). \qquad (137)$$

For simplicity of the representation of the Fourier transform, we discuss the zero-temperature case for a while. Employing (113) the four-divergence at zero temperature[28] is written as

$$\sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu} \Lambda_\mu^Q(x,y,z) = \frac{\partial}{\partial x_0} G(x,y) \delta^4(z-x) + \frac{\partial}{\partial y_0} G(x,y) \delta^4(z-y), \qquad (138)$$

The Fourier transform of Λ_μ^Q becomes

$$\int d^4 z e^{-ikz} \int d^4 x e^{-ip'x} \int d^4 y e^{ipy} \Lambda_\mu^Q(x,y,z) = \Lambda_\mu^Q(p, p-k)(2\pi)^4 \delta^4(-k-p'+p), \qquad (139)$$

where we have assumed the translational invariance (regarded Λ_μ^Q as a function of $y - x$ and $z - x$) and set $\Lambda_\mu^Q(x,y,z) = \Lambda_\mu^Q(x-y, z-x)$, and

$$\Lambda_\mu^Q(p, p-k) = \int d^4(x-y) e^{-ip(x-y)} \int d^4(z-x) e^{-ik(z-x)} \Lambda_\mu^Q(x-y, z-x). \qquad (140)$$

At the same time we set $G(x,y) = G(x-y)$ and introduce the Fourier transform

$$\frac{\partial}{\partial x_0} G(x-y) = \int \frac{d^4 p'}{(2\pi)^4} e^{ip'(x-y)} i p_0' G(p'). \qquad (141)$$

---

[28] Starting from

$$\Lambda_\mu^Q(x,y,z) = \langle T \{ j_\mu^Q(z) \psi_\uparrow(x) \psi_\uparrow^\dagger(y) \} \rangle,$$

we obtain

$$\sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu} \Lambda_\mu^Q(x,y,z) = \langle T \{ [j_0^Q(z), \psi_\uparrow(x)] \psi_\uparrow^\dagger(y) \} \rangle \delta(z_0 - x_0)$$
$$+ \langle T \{ \psi_\uparrow(x) [j_0^Q(z), \psi_\uparrow^\dagger(y)] \} \rangle \delta(z_0 - y_0).$$

22

Then we obtain

$$\sum_{\mu=0}^{3} k_\mu \Lambda_\mu^Q(p, p - k) = p_0 G(p) - (p_0 - k_0)G(p - k). \tag{142}$$

It should be noted that the factor p_0 or $p_0 - k_0$ represents the energy carried by an electron and is automatically taken into account by the commutation relation. By shifting the four-momentum this relation is converted into the Ward identity

$$\sum_{\mu=0}^{3} k_\mu \Gamma_\mu^Q(p + k, p) = p_0 G(p + k)^{-1} - (p_0 + k_0)G(p)^{-1}, \tag{143}$$

for heat current vertex. Here we have used the same relation $\Lambda_\mu^Q(p + k, p) = iG(p + k) \cdot \Gamma_\mu^Q(p + k, p) \cdot iG(p)$ as (120). This identity also holds at finite temperature.[29]

The extension[30] of this Ward identity to the case of Cooper pairs is also straightforward. The Ward identity for heat current vertex of Cooper pairs is

$$\sum_{\mu=0}^{3} k_\mu \Delta_\mu^Q(q + k, q) = q_0 D(q + k)^{-1} - (q_0 + k_0)D(q)^{-1}, \tag{144}$$

where Δ_μ^Q is the counterpart of Γ_μ^Q. It should be noted that the factor q_0 or $q_0 + k_0$ represents the energy carried by a Cooper pair and is automatically taken into account by the commutation relation.

Finally we discuss an application of the Ward identities. If we substitute the quasi-particle propagators[31]

$$G(p)^{-1} = -p_0 - \xi_{\mathbf{p}-\frac{\mathbf{k}}{2}}, \qquad G(p + k)^{-1} = -p_0 - k_0 - \xi_{\mathbf{p}+\frac{\mathbf{k}}{2}}, \tag{145}$$

for the full propagators in (121), we obtain

$$\sum_{\mu=0}^{3} k_\mu \Gamma_\mu^e = ek_0 + e\big(\xi_{\mathbf{p}+\frac{\mathbf{k}}{2}} - \xi_{\mathbf{p}-\frac{\mathbf{k}}{2}}\big), \tag{146}$$

---

[29] As in the case of the Ward identity for charge-current vertex, the relation

$$\Lambda_\mu^Q(p + k, p) = \big[- G(p + k)\big] \cdot \Gamma_\mu^Q(p + k, p) \cdot \big[- G(p)\big],$$

should be taken into account.

[30] See [Nar].

[31] We take the momenta of $G(p)^{-1}$ and $G(p + k)^{-1}$ symmetrically.

so that for $\mu = 1, 2, 3$

$$\Gamma_\mu^e = \frac{e}{m^*} p_\mu = e v_\mu, \tag{147}$$

where we have employed (4) by substituting the renormalized mass of quasi-particles m^* for m. In the same manner we obtain

$$\sum_{\mu=0}^{3} k_\mu \Gamma_\mu^Q = -p_0 \xi_{\mathbf{p}+\frac{\mathbf{k}}{2}} + (p_0 + k_0) \xi_{\mathbf{p}-\frac{\mathbf{k}}{2}}, \tag{148}$$

from (143). Thus for $\mu = 1, 2, 3$

$$\Gamma_\mu^Q \doteq -\frac{1}{2m^*} p_\mu \big[p_0 + (p_0 + k_0) \big] = v_\mu \Big(i\varepsilon_n + \frac{i\omega_\lambda}{2} \Big). \tag{149}$$

These current vertices (147) and (149) are consistent with (94) and (96) of the Jonson-Mahan formula.

If we substitute the Cooper-pair propagators[32]

$$D(\mathbf{q}, i\omega_m{=}0)^{-1} = D(0,0)^{-1} \Big[1 + \xi^2 \Big(\mathbf{q} - \frac{\mathbf{k}}{2} \Big)^2 \Big], \tag{150}$$

for the full propagators in (133), we obtain

$$\sum_{\mu=1}^{3} k_\mu \Delta_\mu^e = 2e D(0,0)^{-1} \xi^2 \Big[\Big(\mathbf{q} - \frac{\mathbf{k}}{2} \Big)^2 - \Big(\mathbf{q} + \frac{\mathbf{k}}{2} \Big)^2 \Big], \tag{151}$$

so that for $\mu = 1, 2, 3$

$$\Delta_\mu^e = -4e D(0,0)^{-1} \xi^2 q_\mu. \tag{152}$$

In the same manner we obtain

$$\Delta_\mu^Q = D(0,0)^{-1} \xi^2 q_\mu \big[q_0 + (q_0 + k_0) \big] = -2 D(0,0)^{-1} \xi^2 q_\mu \Big(i\omega_m + \frac{i\omega_\lambda}{2} \Big), \tag{153}$$

from (144) for $\mu = 1, 2, 3$. These current vertices (152) and (153) are consistent with (101) and (102) of the Jonson-Mahan formula.

11 Quasi-particle Transport: Relaxation-Time Approximation

Within the relaxation-time approximation we make a microscopic calculation equivalent to the Boltzmann transport in the following.

---

[32] In general this Ornstein-Zernike form is expected in the long-wavelength limit. We take the momenta of $D(q)^{-1}$ and $D(q+k)^{-1}$ symmetrically.
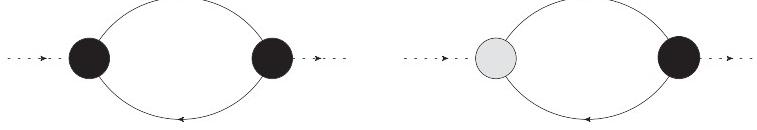
Figure 4: Quasi-particle transport within relaxation-time approximation: The left Feynman diagram describes the charge response to the external electric field. The right describes the heat response to the electric field.

The relaxation-time approximation corresponds to neglecting the vertex correction as shown in Fig. 4. In this approximation Φ^e_{xx} is given by[33]

$$\Phi^e_{xx}(\mathbf{k}=0, i\omega_\lambda) = -\frac{2e^2}{\beta} \sum_{\mathbf{p}} \sum_n v_x^2 G(i\varepsilon_n) G(i\varepsilon_n + i\omega_\lambda), \qquad (154)$$

where the minus sign is the fermion-loop factor. We employ the analytic continuation of the thermal propagator $G(i\varepsilon_n)$ defined for pure imaginary frequencies to the retarded or advanced propagator, $G^R(\epsilon)$ or $G^A(\epsilon)$,

$$G^R(\epsilon) = \frac{1}{\epsilon - \xi_{\mathbf{p}} + i/2\tau}, \qquad G^A(\epsilon) = \frac{1}{\epsilon - \xi_{\mathbf{p}} - i/2\tau}, \qquad (155)$$

for real frequencies. The analytic continuation results in $G^R(\epsilon)$ for $\varepsilon_n > 0$ and $G^A(\epsilon)$ for $\varepsilon_n < 0$. We have to calculate the discrete summation

$$I^e(i\omega_\lambda) \equiv -\frac{1}{\beta} \sum_n G(i\varepsilon_n) G(i\varepsilon_n + i\omega_\lambda). \qquad (156)$$

The summation is transformed into the integral[34] on the contour C in Fig. 5 as

$$I^e(i\omega_\lambda) = \int_C \frac{dz}{2\pi i} f(z) G(z) G(z + i\omega_\lambda). \qquad (157)$$

---

[33] The footnote for (93) shows how (154) results from the Jonson-Mahan formula. Alternatively

$$\Phi^e_{xx}(\mathbf{k}=0, i\omega_\lambda) = -\frac{1}{\beta} \sum_{\mathbf{p}} \sum_n \Gamma^e_x G(i\varepsilon_n) G(i\varepsilon_n + i\omega_\lambda) \Gamma^e_x,$$

results in (154) using the Ward identity (147).

[34] See, for example, §25 and Problem 7.6 in [FW]. Since $\tanh(z/2T) = 1 - 2f(z)$,

$$-I^e(i\omega_\lambda) = \int_C \frac{dz}{4\pi i} \tanh \frac{z}{2T} G(z) G(z + i\omega_\lambda).$$
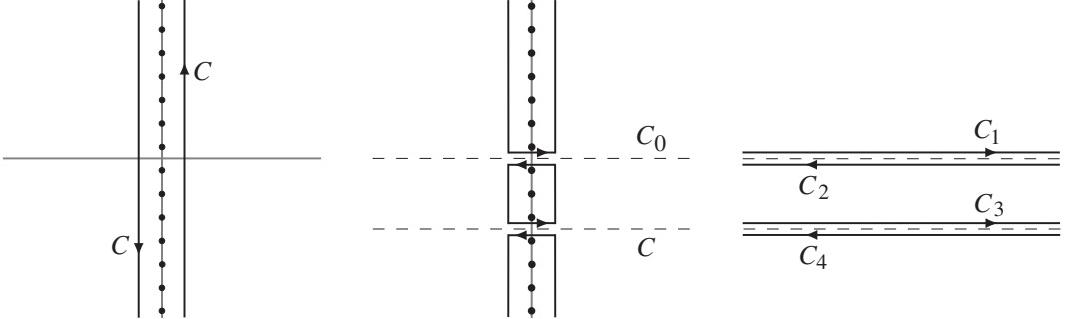
Figure 5: Contours of the integral: These three panels are three different characterization of the same complex z-plane. The solid line with an arrow represents the contour of the integral. The horizontal and vertical gray lines represent the real and imaginary axes of the complex z-plane. The dots on the imaginary axis represent the fermionic thermal frequencies. The broken lines C_0 and C_λ represent the cuts along $\mathrm{Im}\, z = 0$ and $\mathrm{Im}\, z = -\omega_\lambda$.

The contour C is transformed into C_1, C_2, C_3 and C_4 without encountering the singularity[35] of the propagator as shown in Fig. 5. The analytic continuation of $G(i\varepsilon_n)G(i\varepsilon_n + i\omega_\lambda)$ is $G^R(z)G^R(z+i\omega_\lambda)$ on C_1, $G^A(z)G^R(z+i\omega_\lambda)$ on C_2 and C_3 and $G^A(z)G^A(z+i\omega_\lambda)$ on C_4. Taking into account of the direction of four contours we obtain

$$I^e(i\omega_\lambda) = \int_{C_0} \frac{dz}{2\pi i} f(z) G^R(z + i\omega_\lambda)\Big[G^R(z) - G^A(z) \Big]$$
$$+ \int_{C_\lambda} \frac{dz}{2\pi i} f(z) G^A(z) \Big[G^R(z + i\omega_\lambda) - G^A(z + i\omega_\lambda) \Big]. \qquad (158)$$

Shifting the variable in the second integral in (158) we obtain

$$I^e(i\omega_\lambda) = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} f(\epsilon)\Big[G^R(\epsilon + i\omega_\lambda) + G^A(\epsilon - i\omega_\lambda) \Big]\Big[G^R(\epsilon) - G^A(\epsilon) \Big], \quad (159)$$

where we have used the relation $f(\epsilon - i\omega_\lambda) = f(\epsilon)$ led by $e^{-\beta i\omega_\lambda} = 1$ for bosonic frequency ω_λ. Since

$$G^R(\epsilon) - G^A(\epsilon) = 2i\, \mathrm{Im} G^R(\epsilon), \qquad (160)$$

---

[35] The propagator

$$G(z) = \int_{-\infty}^{\infty} d\epsilon \frac{\rho(\epsilon)}{z - \epsilon},$$

has a cut along $\mathrm{Im}\, z = 0$ so that $G(z)G(z + i\omega_\lambda)$ has cuts along $\mathrm{Im}\, z = 0$ and $\mathrm{Im}\, z = -\omega_\lambda$.

we finally obtain

$$I^e(\omega + i\delta) = \int_{-\infty}^{\infty} \frac{d\epsilon}{\pi} f(\epsilon) \Big[G^R(\epsilon + \omega) + G^A(\epsilon - \omega) \Big] \text{Im} G^R(\epsilon). \qquad (161)$$

To calculate the DC conductivity we only need the contribution linear in ω

$$
\begin{aligned}
I^e(\omega + i\delta) - I^e(i\delta) &\doteqdot \omega \int_{-\infty}^{\infty} \frac{d\epsilon}{\pi} f(\epsilon) \left[\frac{\partial G^R(\epsilon)}{\partial \epsilon} - \frac{\partial G^A(\epsilon)}{\partial \epsilon} \right] \text{Im} G^R(\epsilon) \\
&= 2i\omega \int_{-\infty}^{\infty} \frac{d\epsilon}{\pi} f(\epsilon) \left[\frac{\partial}{\partial \epsilon} \text{Im} G^R(\epsilon) \right] \text{Im} G^R(\epsilon) \\
&= i\omega \int_{-\infty}^{\infty} \frac{d\epsilon}{\pi} \left(- \frac{\partial f(\epsilon)}{\partial \epsilon} \right) \Big[\text{Im} G^R(\epsilon) \Big]^2, \qquad (162)
\end{aligned}
$$

where we have employed the integration by parts. Thus the DC conductivity σ_{xx} is given as

$$
\begin{aligned}
\sigma_{xx} &= \lim_{\omega \to 0} \frac{1}{i\omega} \Big[\Phi_{xx}^e(\mathbf{k}{=}0, \omega + i\delta) - \Phi_{xx}^e(\mathbf{k}{=}0, i\delta) \Big] \\
&= 2e^2 \int_{-\infty}^{\infty} \frac{d\epsilon}{\pi} \left(- \frac{\partial f(\epsilon)}{\partial \epsilon} \right) \sum_{\mathbf{p}} v_x^2 \Big[\text{Im} G^R(\epsilon) \Big]^2. \qquad (163)
\end{aligned}
$$

If the life time τ is not so short ($\epsilon_F \tau \gg 1$ where $\epsilon_F = p_F^2/2m$), $\text{Im} G^R(\epsilon)$ behaves as $-\pi\delta(\epsilon - \xi_{\mathbf{p}})$ so that[36]

$$\sigma_{xx} \sim 2e^2 \sum_{\mathbf{p}} \left(- \frac{\partial f(\xi_{\mathbf{p}})}{\partial \xi_{\mathbf{p}}} \right) v_x^2 \int_{-\infty}^{\infty} \frac{d\epsilon}{\pi} \Big[\text{Im} G^R(\epsilon) \Big]^2. \qquad (164)$$

After the integration[37] over ϵ we obtain

$$\sigma_{xx} \sim 2e^2 \sum_{\mathbf{p}} \left(- \frac{\partial f(\xi_{\mathbf{p}})}{\partial \xi_{\mathbf{p}}} \right) v_x^2 \tau, \qquad (165)$$

---

[36] See, for example, §8.1.2 in Mahan: *Many-Particle Physics*, 3rd edition (Kluwer Academic/Plenum, New York, 2000).

[37] Using the definite integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{2a^3},$$

for $a > 0$ we obtain

$$\int_{-\infty}^{\infty} d\epsilon \Big[\text{Im} G^R(\epsilon) \Big]^2 = \int_{-\infty}^{\infty} d\epsilon \left(\frac{1/2\tau}{(\epsilon - \xi_{\mathbf{p}})^2 + (1/2\tau)^2} \right)^2 = \pi\tau.$$

which is equivalent to (40).

In the same manner[38] as (154) Φ_{xx}^Q is given by

$$\Phi_{xx}^Q(\mathbf{k}=0, i\omega_\lambda) = -\frac{2e}{\beta}\sum_{\mathbf{p}}\sum_n v_x^2\Big(i\varepsilon_n + \frac{i\omega_\lambda}{2}\Big)G(i\varepsilon_n)G(i\varepsilon_n + i\omega_\lambda). \quad (166)$$

The discrete summation

$$I^Q(i\omega_\lambda) \equiv -\frac{1}{\beta}\sum_n \Big(i\varepsilon_n + \frac{i\omega_\lambda}{2}\Big)G(i\varepsilon_n)G(i\varepsilon_n + i\omega_\lambda), \quad (167)$$

is transformed into the integral

$$I^Q(\omega + i\delta) = \int_{-\infty}^{\infty}\frac{d\epsilon}{\pi}f(\epsilon)\Big[\Big(\epsilon + \frac{\omega}{2}\Big)G^R(\epsilon + \omega) + \Big(\epsilon - \frac{\omega}{2}\Big)G^A(\epsilon - \omega)\Big]\mathrm{Im}G^R(\epsilon), \quad (168)$$

in the same manner as (161). The ω-linear contribution is evaluated[39] as

$$I^Q(\omega + i\delta) - I^Q(i\delta) \fallingdotseq \omega\int_{-\infty}^{\infty}\frac{d\epsilon}{\pi}f(\epsilon)\Big\{\epsilon\Big[\frac{\partial G^R(\epsilon)}{\partial\epsilon} - \frac{\partial G^A(\epsilon)}{\partial\epsilon}\Big] + \frac{1}{2}\Big[G^R(\epsilon) - G^A(\epsilon)\Big]\Big\}\mathrm{Im}G^R(\epsilon)$$

$$= i\omega\int_{-\infty}^{\infty}\frac{d\epsilon}{\pi}\Big(-\frac{\partial f(\epsilon)}{\partial\epsilon}\Big)\epsilon\Big[\mathrm{Im}G^R(\epsilon)\Big]^2. \quad (169)$$

Thus

$$\tilde{\alpha}_{xx}(\mathbf{k}=0, \omega\to 0) = \lim_{\omega\to 0}\frac{1}{i\omega}\Big[\Phi_{xx}^Q(\mathbf{k}=0, \omega + i\delta) - \Phi_{xx}^Q(\mathbf{k}=0, i\delta)\Big]$$

$$= 2e\int_{-\infty}^{\infty}\frac{d\epsilon}{\pi}\Big(-\frac{\partial f(\epsilon)}{\partial\epsilon}\Big)\epsilon\sum_{\mathbf{p}}v_x^2\Big[\mathrm{Im}G^R(\epsilon)\Big]^2. \quad (170)$$

---

[38] Using (147) and (149)

$$\Phi_{xx}^Q(\mathbf{k}=0, i\omega_\lambda) = -\frac{1}{\beta}\sum_{\mathbf{p}}\sum_n\Gamma_x^Q G(i\varepsilon_n)G(i\varepsilon_n + i\omega_\lambda)\Gamma_x^e,$$

results in (166).

[39] Using (160), the right-hand side of the first line of (169) is equal to

$$i\omega\int_{-\infty}^{\infty}\frac{d\epsilon}{\pi}f(\epsilon)\Big[2\epsilon\frac{\partial}{\partial\epsilon}\mathrm{Im}G^R(\epsilon) + \mathrm{Im}G^R(\epsilon)\Big]\mathrm{Im}G^R(\epsilon) = i\omega\int_{-\infty}^{\infty}\frac{d\epsilon}{\pi}\Big\{-\Big[\frac{\partial}{\partial\epsilon}\big(f(\epsilon)\cdot\epsilon\big)\Big] + f(\epsilon)\Big\}\Big[\mathrm{Im}G^R(\epsilon)\Big]^2,$$

where we have employed the integration by parts.

In the same manner as (164) we obtain[40]

$$\tilde{\alpha}_{xx}(\mathbf{k}=0, \omega \to 0) \sim 2e \sum_{\mathbf{p}} \left(-\frac{\partial f(\xi_{\mathbf{p}})}{\partial \xi_{\mathbf{p}}} \right) v_x^2 \xi_{\mathbf{p}} \tau, \qquad (171)$$

which is equivalent to (41).

As another way of evaluating the summation in (156) we employ the spectral representation (20) and obtain[41]

$$
\begin{aligned}
I^e(i\omega_\lambda) &= -\frac{1}{\beta} \sum_n \int_{-\infty}^{\infty} \frac{\rho(x)dx}{i\varepsilon_n - x} \int_{-\infty}^{\infty} \frac{\rho(y)dy}{i\varepsilon_n + i\omega_\lambda - y} \\
&= -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{\rho(x)\rho(y)}{x - y + i\omega_\lambda} \frac{1}{\beta} \sum_n \left[\frac{1}{i\varepsilon_n - x} - \frac{1}{i\varepsilon_n + i\omega_\lambda - y} \right] \\
&= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{\rho(x)\rho(y)}{x - y + i\omega_\lambda} \big[f(y) - f(x) \big].
\end{aligned}
\qquad (172)
$$

The imaginary part is

$$
\begin{aligned}
\mathrm{Im}\big[I^e(\omega + i\delta) \big] &= -\pi \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \rho(x)\rho(y) \big[f(y) - f(x) \big] \delta(x - y + \omega) \\
&= -\pi \int_{-\infty}^{\infty} dx \rho(x)\rho(x + \omega) \big[f(x + \omega) - f(x) \big] \\
&= -\pi \int_{-\infty}^{\infty} dx \rho(x) \big[\rho(x - \omega) - \rho(x + \omega) \big] f(x).
\end{aligned}
\qquad (173)
$$

The ω-linear contribution is estimated as

$$\mathrm{Im}\big[I^e(\omega + i\delta) \big] \fallingdotseq \pi \int_{-\infty}^{\infty} dx \rho(x) \frac{\partial \rho(x)}{\partial x} \cdot 2\omega \cdot f(x). \qquad (174)$$

Employing the integration by parts we obtain the final result equivalent to (162)

$$\mathrm{Im}\big[I^e(\omega + i\delta) \big] \fallingdotseq \pi \omega \int_{-\infty}^{\infty} dx \big[\rho(x) \big]^2 \left(-\frac{\partial f(x)}{\partial x} \right). \qquad (175)$$

---

[40] Formally $\tilde{\alpha}_{xx}$ is proportional to the charge of electrons. However, the sign of the summation at low temperature is determined by that of N_1 where the density of states $N(\xi_{\mathbf{p}})$ is expanded as $N(\xi_{\mathbf{p}}) \fallingdotseq N(0) + N_1 \cdot \xi_{\mathbf{p}}$. In the presence of particle-hole symmetry, N_1 and thus $\tilde{\alpha}_{xx}$ vanish. See (10.19) in [2] for the discussion on the symmetry.

[41] Here the relation

$$\lim_{\delta \to 0} \frac{1}{\beta} \sum_n \frac{e^{i\varepsilon_n \delta}}{i\varepsilon_n - x} = f(x),$$

is employed taking into account that $\varepsilon_n + \omega_\lambda$ is also a fermionic thermal frequency. See, for example, §25 in [FW].

12　GL Transport: Gaussian Fluctuation

Here we discuss the GL transport.[42] We consider the GL free-energy density

$$E(\mathbf{x}) = \alpha \big|\Psi(\mathbf{x})\big|^2 + \frac{1}{2m^*}\bigg|\frac{\nabla}{i}\Psi(\mathbf{x})\bigg|^2, \tag{176}$$

in the Gaussian approximation. Introducing the Fourier transform of the complex order-parameter field

$$\Psi(\mathbf{x}) = \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}\Psi_{\mathbf{q}}, \qquad \Psi^*(\mathbf{x}) = \sum_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{x}}\Psi_{\mathbf{q}}^*, \tag{177}$$

the GL free-energy E is given by

$$E = \int d^3x\, E(\mathbf{x}) = \sum_{\mathbf{q}} \Big(\alpha + \frac{q^2}{2m^*}\Big)\Psi_{\mathbf{q}}^*\Psi_{\mathbf{q}}. \tag{178}$$

The Gaussian fluctuation is evaluated as[43]

$$\big\langle\Psi_{\mathbf{q}}^*\Psi_{\mathbf{q}}\big\rangle = \frac{T}{\alpha + q^2/2m^*} = \frac{T}{\alpha}\frac{1}{1+\xi^2q^2}, \tag{179}$$

where $\langle A\rangle$ is the expectation value of A under the Boltzmann weight $\exp(-E/T)$. Here the correlation length ξ of the fluctuation is introduced as

$$\xi^2 = \frac{1}{2m^*\alpha}. \tag{180}$$

The relaxation of the fluctuation is determined by the time-dependent GL equation

$$\gamma\frac{\partial\Psi_{\mathbf{q}}}{\partial t} = -\frac{\delta E}{\delta\Psi_{\mathbf{q}}^*}. \tag{181}$$

In terms of the relaxation time $\tau_{\mathbf{q}}$

$$\frac{1}{\tau_{\mathbf{q}}} = \frac{1}{\gamma}\Big(\alpha + \frac{q^2}{2m^*}\Big) = \frac{\alpha}{\gamma}\big(1+\xi^2q^2\big), \tag{182}$$

---

[42] We follow the discussion for the conductivity by Skocpol and Tinkham: Rep. Prog. Phys. **38**, 1049 (1975) and Tinkham: *Introduction to Superconductivity*, 2nd edition (McGraw-Hill, New York, 1996). Then we extend the discussion to the case of the thermoelectric tensor and our result (197) is identical to that in [3] and [5].

[43] The Gauss integral for complex variable z is performed as

$$\int \frac{dz^*dz}{2\pi i}e^{-az^*z} = \int \frac{dudv}{\pi}e^{-a(u^2+v^2)} = \frac{1}{a}.$$

(181) is written as

$$\frac{\partial \Psi_{\mathbf{q}}}{\partial t} = -\frac{1}{\tau_{\mathbf{q}}} \Psi_{\mathbf{q}}. \tag{183}$$

The Kubo formula for classical variables is

$$\sigma_{xx} = \frac{1}{T} \int_0^\infty \langle J_x^e(0) J_x^e(t) \rangle dt, \tag{184}$$

where the charge current[44] is given by

$$J_x^e(t) = \frac{e^*}{m^*} \sum_{\mathbf{q}} q_x \Psi_{\mathbf{q}}^*(t) \Psi_{\mathbf{q}}(t). \tag{185}$$

The current-current correlation

$$\langle J_x^e(0) J_x^e(t) \rangle = \left(\frac{e^*}{m^*}\right)^2 \sum_{\mathbf{q}} q_x^2 \langle |\Psi_{\mathbf{q}}^*(0) \Psi_{\mathbf{q}}(t)|^2 \rangle, \tag{186}$$

is evaluated by using

$$\langle |\Psi_{\mathbf{q}}^*(0) \Psi_{\mathbf{q}}(t)|^2 \rangle = \langle \Psi_{\mathbf{q}}^* \Psi_{\mathbf{q}} \rangle^2 e^{-2t/\tau_{\mathbf{q}}}, \tag{187}$$

where the time-dependence of the order-parameter field is determined by (183) as

$$\Psi_{\mathbf{q}}(t) = \Psi_{\mathbf{q}} e^{-t/\tau_{\mathbf{q}}}. \tag{188}$$

After the time-integral we obtain

$$\sigma_{xx} = \left(\frac{e^*}{m^*}\right)^2 \frac{1}{T} \sum_{\mathbf{q}} q_x^2 \langle \Psi_{\mathbf{q}}^* \Psi_{\mathbf{q}} \rangle^2 \frac{\tau_{\mathbf{q}}}{2}. \tag{189}$$

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[44] The charge and heat currents corresponding to (97) and (98) are given by

$$\mathbf{j}^e(\mathbf{x}) = \frac{e^*}{2m^*i} \left[\Psi^*(\mathbf{x}) \left(\nabla \Psi(\mathbf{x})\right) - \left(\nabla \Psi^*(\mathbf{x})\right) \Psi(\mathbf{x}) \right],$$

and

$$\mathbf{j}^Q(\mathbf{x}) = -\frac{1}{2m^*} \left[\left(\nabla \Psi^*(\mathbf{x})\right) \dot{\Psi}(\mathbf{x}) + \dot{\Psi}^*(\mathbf{x}) \left(\nabla \Psi(\mathbf{x})\right) \right],$$

as (4) and (5) in [5]. Using (177) the Fourier-transformed charge current is

$$\mathbf{j}^e(\mathbf{k}) = \frac{e^*}{m^*} \sum_{\mathbf{q}} \left(\mathbf{q} + \frac{\mathbf{k}}{2}\right) \Psi_{\mathbf{q}}^* \Psi_{\mathbf{q}+\mathbf{k}},$$

as (73). Thus the x-component of the uniform current $J_x^e = j_x^e(\mathbf{k}{=}0)$ is given by (185).

Using (179) we obtain the DC conductivity

$$\sigma_{xx} = \pi e^2 \frac{T}{T - T_c} \sum_{\mathbf{q}} \frac{\xi^4 q_x^2}{(1 + \xi^2 q^2)^3}. \tag{190}$$

In 2D the $\mathbf{q}$-summation is performed as[45]

$$\sum_{\mathbf{q}} \frac{\xi^4 q_x^2}{(1 + \xi^2 q^2)^3} = \frac{1}{8\pi d} \int_0^\infty dx \frac{x}{(x + 1)^3} = \frac{1}{16\pi d}, \tag{191}$$

so that

$$\sigma_{xx} = \frac{e^2}{16d} \frac{T}{T - T_c}. \tag{192}$$

Next we consider the Kubo formula for the thermo-electric tensor $\tilde{\alpha}$

$$\tilde{\alpha}_{xx} = \frac{1}{T} \int_0^\infty \left\langle J_x^e(0) J_x^Q(t) \right\rangle dt, \tag{193}$$

where the heat current is given by

$$J_x^Q(t) = -\frac{i}{2m^*} \lim_{t' \to t} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right) \sum_{\mathbf{q}} q_x \Psi_{\mathbf{q}}^*(t) \Psi_{\mathbf{q}}(t'), \tag{194}$$

which corresponds to (86). Although in (188) $\tau_{\mathbf{q}}$ is assumed to be real,[46] here it is extended to be complex[47] and using its complex-conjugate $\tau_{\mathbf{q}}^*$ the

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[45] Here we consider the film whose thickness is d and assume $d \ll \xi$ so that

$$\sum_{\mathbf{q}} \to \frac{1}{2\pi d} \int_0^\infty q \, dq.$$

The replacement $q_x^2 \to q^2/2$ can be done for the present symmetric case in 2D. Since

$$\int dx \frac{x}{(x + 1)^3} = \int dx \frac{1}{(x + 1)^2} - \int dx \frac{1}{(x + 1)^3},$$

we obtain

$$\int_0^\infty dx \frac{x}{(x + 1)^3} = \frac{1}{2}.$$

[46] If $\tau_{\mathbf{q}}$ is real, α_{xx} vanishes. Since holes carry opposite charge to electrons, α_{xx} and σ_{xy} that are odd in the electric charge vanish by the cancelation between contributions from particle- and hole- charge currents in the presence of the particle-hole symmetry. On the other hand, such a cancelation does not work for the heat current. Even in the presence of the particle-hole symmetry σ_{xx} and α_{xy} that are even in the electric charge do not vanish. See, for example, Niven and Smith: Phys. Rev. B **66**, 214505 (2002) for the consideration in terms of the symmetry.

[47] In general the complex GL equation

$$\frac{\partial \tilde{\Psi}}{\partial t} = \tilde{\Psi} + (1 + ic_1)\nabla^2 \tilde{\Psi} - (1 + ic_2)|\tilde{\Psi}|^2 \tilde{\Psi},$$

time-dependence of the conjugate of (188) is given as

$$\Psi_{\mathbf{q}}^{*}(t) = \Psi_{\mathbf{q}}^{*} e^{-t/\tau_{\mathbf{q}}^{*}}. \tag{195}$$

Since

$$-\frac{i}{2}\lim_{t' \to t}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)\Psi_{\mathbf{q}}^{*}(t)\Psi_{\mathbf{q}}(t') = -\frac{\tau_{\mathbf{q}}^{''}}{(\tau_{\mathbf{q}}^{'})^2 + (\tau_{\mathbf{q}}^{''})^2}\Psi_{\mathbf{q}}^{*}(t)\Psi_{\mathbf{q}}(t), \tag{196}$$

we obtain[48] within the linear order[49] of $\gamma^{''}/\gamma^{'}$

$$\tilde{\alpha}_{xx} \fallingdotseq \frac{e^{*}}{(m^{*})^2}\frac{1}{T}\sum_{\mathbf{q}} q_x^2 \langle \Psi_{\mathbf{q}}^{*}\Psi_{\mathbf{q}}\rangle^2 \frac{\tau_{\mathbf{q}}^{'}}{2} \times \left(-\frac{\gamma^{''}}{\gamma^{'}}\right)\frac{1}{\tau_{\mathbf{q}}^{'}}$$

$$= -2e^{*}T\frac{\gamma^{''}}{\gamma^{'}}\sum_{\mathbf{q}}\frac{\xi^4 q_x^2}{(1 + \xi^2 q^2)^2}, \tag{197}$$

where we have assumed $\gamma^{'} \gg |\gamma^{''}|$ with $\gamma \equiv \gamma^{'} + i\gamma^{''}$ and used (180).

In 2D the $\mathbf{q}$-summation is performed as[50]

$$\sum_{\mathbf{q}}\frac{\xi^4 q_x^2}{(1 + \xi^2 q^2)^2} \fallingdotseq \frac{1}{8\pi d}\int_0^{q_c^2} dx \frac{x}{(x + \xi^{-2})^2} \fallingdotseq \frac{1}{8\pi d}\ln(q_c\xi)^2, \tag{198}$$

so that[51]

$$\alpha_{xx} \fallingdotseq \frac{|e|}{2\pi d}\frac{\gamma^{''}}{\gamma^{'}}\ln\frac{T_\Lambda}{T - T_c}, \tag{199}$$

where we have used the Onsager relation (77) and introduced T_Λ as $(q_c\xi)^2 \equiv T_\Lambda/(T - T_c)$.
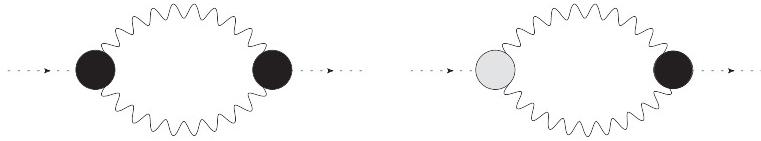
Figure 6: AL process of Cooper-pair transport.

13 Cooper-Pair Transport: AL Process

Here we derive the GL results (192) and (199) from the AL process of the microscopic calculation. Such a process corresponds to the relaxation-time approximation neglecting the vertex correction. Since the effective interaction L also obeys the same relations[52] as (133) and (144), we obtain the

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is employed to study the dynamics of the complex order parameter $\tilde{\Psi}$. The origins of the imaginary part of $\tau_{\mathbf{q}}$ are discussed in §9.3 of [3] for example. The non-vanishing imaginary part is related to the violation of the particle-hole symmetry.

[48] The Drude-like formula (189) was extended to the case of the thermo-electric tensor by Howson, Salamon, Friedmann, Rice and Ginsberg: Phys. Rev. B **41**, 300 (1990) and Ausloos, Clippe and Patapis: Phys. Rev. B **46**, 5763 (1992). However, their discussion is incorrect so that they could not reach (197).

[49] We have put $\tau_{\mathbf{q}}'' = 0$ except the numerator of the right-hand side of (196). Thus the time-integral is identical to that leads to (189).

[50]

$$\int dx \frac{x}{(x+b)^2} = \int dx \frac{1}{x+b} - b \int dx \frac{1}{(x+b)^2}.$$

[51] Our result (199) is identical to (4.31) in [3] and TABLE I in [5].

[52] The Ward identities for L are given as

$$\sum_{\mu=0}^{3} k_\mu \tilde{\Delta}_\mu^e(q,k) = 2eL(q)^{-1} - 2eL(q+k)^{-1},$$

and

$$\sum_{\mu=0}^{3} k_\mu \tilde{\Delta}_\mu^Q(q+k,q) = q_0 L(q+k)^{-1} - (q_0+k_0)L(q)^{-1},$$

and we employ

$$L(\mathbf{q}, i\omega_m = 0)^{-1} = -N(0)\left[\epsilon + \xi_0^2\left(\mathbf{q} - \frac{\mathbf{k}}{2}\right)^2\right],$$

as (150).

current vertices[53] for L as

$$\tilde{\Delta}_\mu^e = 4eN(0)\xi_0^2 q_\mu. \tag{200}$$

and

$$\tilde{\Delta}_\mu^Q = 2N(0)\xi_0^2 q_\mu\left(i\omega_m + \frac{i\omega_\lambda}{2}\right). \tag{201}$$

These relations also hold even when we take the effect of impurity scattering into account.[54]

The charge response to electric field is determined by

$$\Phi_{xx}^e(\mathbf{k}{=}0, i\omega_\lambda) = \frac{1}{\beta}\sum_{\mathbf{q}}\sum_m \tilde{\Delta}_x^e L(i\omega_m)L(i\omega_m + i\omega_\lambda)\tilde{\Delta}_x^e, \tag{202}$$

where we have used the simplified notation $L(i\omega_m) \equiv L(\mathbf{q}, i\omega_m)$. The discrete summation

$$I^e(i\omega_\lambda) \equiv \frac{1}{\beta}\sum_m L(i\omega_m)L(i\omega_m + i\omega_\lambda), \tag{203}$$

is transformed into the integral[55]

$$I^e(i\omega_\lambda) = \int_C \frac{dz}{2\pi i}n(z)L(z)L(z + i\omega_\lambda), \tag{204}$$

where the contour C is shown in Fig. 5 and

$$n(z) = \frac{1}{e^{\beta z} - 1}, \tag{205}$$

is the Bose distribution function. Employing

$$L^R(x) - L^A(x) = 2i\,\mathrm{Im}L^R(x), \tag{206}$$

and the relation $n(z - i\omega_\lambda) = n(z)$ led by $e^{-i\beta\omega_\lambda} = 1$ we obtain

$$I^e(\omega + i\delta) = \int_{-\infty}^{\infty}\frac{dx}{\pi}n(x)\Big[L^R(x + \omega) + L^A(x - \omega)\Big]\mathrm{Im}L^R(x), \tag{207}$$

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[53] The frequency factor in (201) appeared in (34) of [Uss]. However, such a factor cannot be derived from the preceding discussion in [Uss]. I shall discuss this point in the supplement noticed in the footnote 63.

[Uss] $\equiv$ Ussishkin: Phys. Rev. B **68**, 024517 (2003).

[54] See, for example, §8.3 in [3]. The value of ξ_0 becomes a function of the mean free path due to impurity scattering.

[55] See, for example, §25 in [FW]. Since $\coth(z/2T) = 1 + 2n(z)$,

$$I^e(i\omega_\lambda) = \int_C \frac{dz}{4\pi i}\coth\frac{z}{2T}L(z)L(z + i\omega_\lambda).$$

by repeating the procedure leading to (161). The contribution linear in ω is obtained as (162):

$$I^e(\omega + i\delta) - I^e(i\delta) \fallingdotseq i\omega \int_{-\infty}^{\infty} \frac{dx}{\pi} \left(-\frac{\partial n(x)}{\partial x} \right) \left[\mathrm{Im}L^R(x) \right]^2, \qquad (208)$$

where

$$\mathrm{Im}L^R(x) = -\frac{1}{N(0)} \frac{\tau_0 x}{\eta^2 + (\tau_0 x)^2}, \qquad (209)$$

with $\eta \equiv \epsilon + \xi_0^2 \mathbf{q}^2$. Thus the resulting DC conductivity is

$$\sigma_{xx} = \sum_{\mathbf{q}} \left(\tilde{\Delta}_x^e \right)^2 \int_{-\infty}^{\infty} \frac{dx}{\pi} \left(-\frac{\partial n(x)}{\partial x} \right) \left[\mathrm{Im}L^R(x) \right]^2$$

$$= \frac{1}{4\pi T} \sum_{\mathbf{q}} \left(\tilde{\Delta}_x^e \right)^2 \int_{-\infty}^{\infty} dx \left[\sinh \frac{x}{2T} \right]^{-2} \left[\mathrm{Im}L^R(x) \right]^2. \qquad (210)$$

In order to discuss the effect of the low-energy critical fluctuation at finite temperature it is enough to use the high-temperature expansion[56] $\sinh(x/2T) \fallingdotseq x/2T$ under the assumption $\epsilon \ll 1$. By performing the integral[57] we obtain

$$\sigma_{xx} = 8e^2 \tau_0 T \sum_{\mathbf{q}} \frac{\xi_0^4 q_x^2}{(\epsilon + \xi_0^2 \mathbf{q}^2)^3}. \qquad (211)$$

In 2D the $\mathbf{q}$-summation is performed as[58]

$$\sum_{\mathbf{q}} \frac{\xi_0^4 q_x^2}{(\epsilon + \xi_0^2 \mathbf{q}^2)^3} = \frac{1}{8\pi d} \int_0^{\infty} dx \frac{x}{(x+\epsilon)^3} = \frac{1}{16\pi d} \frac{1}{\epsilon}, \qquad (212)$$

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[56] The high-temperature expansion is compatible with the Kubo formula (184) and (193) for classical variables.

[57] Since

$$\left[\sinh \frac{x}{2T} \right]^{-2} \left[\mathrm{Im}L^R(x) \right]^2 \fallingdotseq \left(\frac{2T}{N(0)} \right)^2 \left(\frac{\tau_0}{\eta^2 + (\tau_0 x)^2} \right)^2,$$

the integral is evaluated as

$$\int_{-\infty}^{\infty} dx \left[\sinh \frac{x}{2T} \right]^{-2} \left[\mathrm{Im}L^R(x) \right]^2 \fallingdotseq \left(\frac{2T}{N(0)} \right)^2 2\tau_0 \int_0^{\infty} \frac{dy}{(y^2 + \eta^2)^2}.$$

Using the definite integral appeared in the footnote for (165) we obtain

$$\int_0^{\infty} \frac{dy}{(y^2 + \eta^2)^2} = \frac{\pi}{4} \frac{1}{\eta^3}.$$

[58] By the same procedure as the footnote for (191)

$$\int_0^{\infty} dx \frac{x}{(x+\epsilon)^3} = \frac{1}{2\epsilon}.$$

so that[59]

$$\sigma_{xx} = \frac{e^2}{2\pi d}\frac{\tau_0 T}{\epsilon} = \frac{e^2}{16d}\frac{1}{\epsilon}. \tag{213}$$

The heat response to electric field is determined by

$$\Phi_{xx}^Q(\mathbf{k}=0, i\omega_\lambda) = \frac{1}{\beta}\sum_{\mathbf{q}}\sum_m \tilde{\Delta}_x^Q L(i\omega_m)L(i\omega_m + i\omega_\lambda)\tilde{\Delta}_x^e. \tag{214}$$

The discrete summation

$$I^Q(i\omega_\lambda) \equiv \frac{1}{\beta}\sum_m \left(i\omega_m + \frac{i\omega_\lambda}{2}\right)L(i\omega_m)L(i\omega_m + i\omega_\lambda), \tag{215}$$

is transformed into the integral

$$I^Q(\omega + i\delta) = \int_{-\infty}^{\infty}\frac{dx}{\pi}n(x)\left[\left(x + \frac{\omega}{2}\right)L^R(x + \omega) + \left(x - \frac{\omega}{2}\right)L^A(x - \omega)\right]\mathrm{Im}L^R(x). \tag{216}$$

In the same manner as (169) we obtain

$$I^Q(\omega + i\delta) - I^Q(i\delta) \fallingdotseq i\omega\int_{-\infty}^{\infty}\frac{dx}{\pi}\left(-\frac{\partial n(x)}{\partial x}\right)x\left[\mathrm{Im}L^R(x)\right]^2. \tag{217}$$

Employing the high-temperature expansion $n(x) \fallingdotseq T/x$, (217) reduces to

$$I^Q(\omega + i\delta) - I^Q(i\delta) \fallingdotseq \frac{\omega T}{4\pi}\int_{-\infty}^{\infty}dx\left[L^R(x) - L^A(x)\right]\frac{L^R(x) - L^A(x)}{ix}. \tag{218}$$

Here

$$L^R(x) = -\frac{1}{N(0)}\frac{1}{\eta - i\tau_0 x}, \qquad L^A(x) = -\frac{1}{N(0)}\frac{1}{\eta + i\tau_0^* x}, \tag{219}$$

with $\tau_0 \equiv \tau_1 + i\tau_2$ and $\tau_0^* \equiv \tau_1 - i\tau_2$ and we can put $\tau_1 \gg |\tau_2|$. The right-hand

[59] Here we have used $\tau_0 = \pi/8T$ derived within the ladder approximation. See, for example, §6.2 in [3].

side of (218) is evaluated exactly by the residue as[60]

$$-i\omega\frac{T}{2N(0)^2}\frac{\tau_2}{\tau_1}\frac{1}{\eta^2}. \tag{220}$$

Thus the analytic continuation of (214) is given as[61]

$$\Phi_{xx}^Q(\mathbf{k}=0,\omega+i\delta)=(-i\omega)4eT\frac{\tau_2}{\tau_1}\sum_{\mathbf{q}}\frac{\xi_0^4 q_x^2}{(\epsilon+\xi_0^2\mathbf{q}^2)^2}. \tag{221}$$

In 2D the $\mathbf{q}$-summation is performed as (198):

$$\sum_{\mathbf{q}}\frac{\xi_0^4 q_x^2}{(\epsilon+\xi_0^2\mathbf{q}^2)^2}\doteqdot\frac{1}{8\pi d}\int_0^{x_c}dx\frac{x}{(x+\epsilon)^2}\doteqdot\frac{1}{8\pi d}\ln\frac{x_c}{\epsilon}, \tag{222}$$

so that

$$\tilde{\alpha}_{xx}\doteqdot-\frac{e}{2\pi d}T\frac{\tau_2}{\tau_1}\ln\frac{x_c}{\epsilon}. \tag{223}$$

By the Onsager relation (77) we finally obtain

$$\alpha_{xx}=\frac{1}{T}\tilde{\alpha}_{xx}\doteqdot\frac{|e|}{2\pi d}\frac{\tau_2}{\tau_1}\ln\frac{T_\Lambda}{T-T_c}, \tag{224}$$

which is equivalent to (199) where $\tau_2/\tau_1=\gamma''/\gamma'$ and x_c is chosen as $x_c/\epsilon\equiv T_\Lambda/(T-T_c)$.

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[60] The integral

$$I\equiv\int_{-\infty}^{\infty}dx\Big[L^R(x)-L^A(x)\Big]\frac{L^R(x)-L^A(x)}{ix},$$

is written as

$$I=\frac{\tau_0^*+\tau_0}{N(0)^2}\int_{-\infty}^{\infty}dx\Big[\frac{1}{\eta-i\tau_0 x}-\frac{1}{\eta+i\tau_0^* x}\Big]\frac{1}{(\eta-i\tau_0 x)(\eta+i\tau_0^* x)},$$

and is evaluated by the residue as

$$I=\frac{2\pi}{N(0)^2}\frac{\tau_0^*-\tau_0}{\tau_0^*+\tau_0}\frac{1}{\eta^2},$$

where the position of the pole is determined by the condition $\tau_1\gg|\tau_2|$.

[61] Since $\tau_2<0$ in 3D for our free electron dispersion (4), (221) implies that $\tilde{\alpha}_{xx}$ is proportional to the charge of Cooper pairs. For example, (2.27) in Fukuyama, Ebisawa and Tsuzuki: Prog. Thoer. Phys. **46**, 1028 (1971) shows $\tau_2<0$ after the correction pointed out by several authors: its right-hand side should be multiplied by -1.

14 Ward Identities: Ladder Approximation

It is instructive to show how the Ward identity is satisfied in the ladder approximation that is usually employed to obtain the explicit form of the Cooper-pair propagator. In particular it is very important to clarify how the Ward identity for heat current vertex is satisfied, because it will clear away some confusions[62] seen in the literatures. Such a clarification becomes inevitably complicated so that it will be uploaded as a separate supplementary note.[63]

15 Current Vertices: Perturbational Analysis

As has been shown in this note we do not have to calculate the current vertices for Cooper pairs to obtain σ_{xx} and α_{xx}. However, it is instructive to construct the current vertices by perturbational calculation.[64] Such a construction for the heat current vertex is complicated[65] so that it will be included in the supplementary note noticed in the previous section.

16 Exercise

Find and correct the errors in this note. (Do not trust the results in this note before you check them.)

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[62] For example, if we employ the heat current vertex for electrons with the frequency factor $i\varepsilon_n + i\omega_\lambda/2$ in perturbation expansion, we obtain some wrong results. The frequency factor should be accompanied by the full propagator. It should not be accompanied by the free propagator.

[63] I shall upload a note entitled *A Diagrammer's Note on Superconducting Fluctuation Transport for Beginners: Supplement* besides the three notes in series.

[64] If we perform the perturbational calculation of the current vertex without the ground as the Ward identity, it is difficult to convince ourselves of the validness of the result. For example, there are many errors in the calculation of vertices by Varlamov and Livanov: Sov. Phys. JETP **71**, 325 (1990) and **72**, 1016 (1991), besides the fatal error, concerning the "cancelation", corrected by [RS].

[RS] ≡ Reizer and Sergeev: Phys. Rev. B **50**, 9344 (1994).

[65] The factor of the heat current vertex (45) in [RS] is incorrect but this result is supported by Serbyn, Skvortsov, Varlamov and Galitski: Phys. Rev. Lett. **102**, 067001 (2009). I do not understand the reason of the support. On the other hand, the heat current vertex (21) in [Uss], which is cited as (10.23) in [2] and corresponds to (201) with $\omega_\lambda = 0$, is insufficient to obtain α_{xx}.

Acknowledgements

References

[1] In the following I only list the references that you must read. Neither originality nor priority is considered here. Other references are cited in the footnotes.

[2] Larkin and Varlamov: *Theory of Fluctuations in Superconductors* (Oxford Univ. Press, Oxford, 2005).[66]

[3] Larkin and Varlamov: *Theory of Fluctuations in Superconductors*, revised edition (Oxford Univ. Press, Oxford, 2009).

[4] Schrieffer: *Theory of Superconductivity* (Benjamin-Cummings, Massachusetts, 1964).

[5] Ussishkin, Sondhi and Huse: Phys. Rev. Lett. **89**, 287001 (2002).

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[66] Although we mainly refer the following revised edition, the microscopic calculation for thermal transport is discussed only in this first edition.